Feynman graph representation of the perturbation series for general functional measures

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Abstract. A representation of the perturbation series of a general functional measure is given in terms of generalized Feynman graphs and -rules. The graphical calculus is applied to certain functional measures of Lévy type. A graphical notion of Wick ordering is introduced and is compared with orthogonal decompositions of the Wiener-Itô-Segal type. It is also shown that the linked cluster theorem for Feynman graphs extends to generalized Feynman graphs. We perturbatively prove existence of the thermodynamic limit for the free energy density and the moment functions. The results are applied to the gas of charged microscopic or mesoscopic particles – neutral in average – in $d = 2$ dimensions generating a static field ϕ with quadratic energy density giving rise to a pair interaction. The pressure function for this system is calculated up to fourth order. We also discuss the subtraction of logarithmically divergent self-energy terms for a gas of only one particle type by a local counterterm of first order.

Key words: *Feynman graphs and rules for general functional measures, Wick ordering, linked cluster theorem, free energy density, gas of charged particles.*

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Contents

1 Introduction

Let X, Y be two real random variables such that their joint distribution has a unique solution of the moment problem and $\langle\cdot\rangle$ the expectation value. Then X and Y are independent, if and only if $\langle X^n Y^m\rangle = \langle X^n\rangle\langle Y^m\rangle\ \forall n, m \in \mathbb{N}$. On the left hand side of this equation there is one moment, but on the right hand side there is a product of two moments. This "non-linearity of independence" expressed in terms of moments seems harmless, but it has notable consequences in classical statistical physics, where X and Y have to be replaced by correlated random variables $\phi(x)$ and $\phi(y)$ for x, y in some discrete or continuous position space, and the independence is only asymptotic if the distance between x and y goes to infinity. The matheamtical formulationis that the translation group acts ergodically on the L^2-spaceof the underlying measure or, with a little more physical flavour, that the statistical system under consideration is a pure phase.

The "non-linearity" described above in many cases of interest leads to a rather involved formulae for the moment functions $\langle\phi(x_1)\cdots\phi(x_n)\rangle$. The asymptotic independence can however be "linearized" by passing through a combinatorial procedure to truncated moment functions that fulfill $\langle\phi(x_1)\cdots\phi(x_n)\rangle^T \to 0$ if the separation of the arguments $x_1, \ldots, x_n$ becomes large.

This basic principle is mostly used in calculations, where the asymptotic independence is decisive, like in practically all problems connected with the thermodynamic (TD) limit. In particular this applies to perturbative expansions, where often a sufficiently fast decrease of the truncated functions is all what one needs to carry out the TD limit order by order and to calculate low orders explicitly. Quite often, it is convenient to use graphs to keep track of all the terms that appear in the expansions. A number of excellent textbooks are available on this by now classical topic, see [6, 10, 19, 20, 21, 22] to cite only a few.

In modern texts on the subject, the combinatorial structure of these expansions has been distilled into the notion of abstract polymer system, which is sufficiently flexible to be applied in most classical situations, like spin systems, systems of particles in the continuum and Euclidean quantum field theory. The handling of this concept however depends on the physical situation, where some insight is needed to find out what the polymers are and what is the activity function. While this is satisfactory from the point of view of the given application, conceptually it is somehow less clear.

In this article, we give a perturbative high temperature expansion for the moment functions and the free energy density of a large class of systems of statistical physics, containing in particular the ones named above, that is to a large extent independent of

the nature of the unperturbed system under consideration and works for a large class of interactions. The expansion is only based on the elementary combinatorics of "truncation" and hence the fundamental feature of (asymptotic) independence. The motivation mainly stems from the Feynman graph calculus in perturbative Euclidean quantum field theory (EQFT), see e.g. [10, 14, 21], which we generalize from Gaussian to arbitrary functional measures using Feynman graphs with two kinds ("empty" and "full") of vertices. Full vertices are the known interaction vertices whereas empty vertices with n legs simply symbolize a truncated n-point function.

The article is organized as follows: Basic notations are collected in Section 2 and the perturbation series is introduced. In Section 3 we develop our generalized Feynman graph calculus, which we apply in Section 4 to some measures of Lévy type that have relations to particle systems and quantum field theory, see the references [1–5], containing Gaussian Euclidean quantum field theory as a special case. In fact, for this more general class of models the Feynman rules are particularly simple, just as in the Gaussian case. In Section 5 we introduce a general and measure independent definition of Wick-ordering that is based on the graphical notion of self-contraction. It coincides with orthogonal decompositions of the Wiener-Itô-Segal type [10, 12, 23] if and only if the underlying measure is Gaussian. It is also shown that Wick-ordering removes ultra-violet divergences in $d = 2$ dimensions for a class of models [3] containing also certain fields of Lévy type. The linked cluster theorem for generalized Feynman graphs is the topic of Section 6, where we give a proof which is only based on the combinatorics of truncation. We apply this result to prove the existence of the TD limit of the free energy density in perturbation theory. It is rather simple to extend the results to the TD limit of moment functions using a Schwinger term, which is done in Section 7. In Section 8 we finally apply the results of Section 6 to some particle systems in the continuum – microscopic and mesoscopic – where the number of graphs is very effectively reduced. We consider a gas of charged particles that is neutral in average and interacts via a ϕ^2 energy density of the static field generated by the particles. The pressure function for this system in $d = 2$ dimensions is calculated up to 4th order. Even though the topic of ultra-violet divergences and renormalization to a large extent is beyond the scope of this article, we sketch the renormalization of the perturbation series by a local counterterm for a gas with only one type of particle and logarithmic self-energy divergences, which to some extent is similar to Gaussian ϕ^4-theory in $d = 3$ dimensions.

2 Perturbation series for general functional measures

Let $d \in \mathbb{N}$ be the dimension of the underlying space[1] $\mathbb{R}^d$ (space-time in EQFT). Let ν be a probability measure on the measurable space $(\mathcal{S}', \mathcal{B})$, where $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions and $\mathcal{B} = \mathcal{B}(\mathcal{S}')$ the Borel σ-ring generated by the open sets of the weak topology on $\mathcal{S}'$. For $F : \mathcal{S}' \to \mathbb{R}$ or $F : \mathcal{S}' \to \mathbb{C}$ ν_0-integrable, we set

[1] Obviously, most of the considerations of this article remain valid if one replaces $\mathbb{R}^d$ and the Lebesgue measure dx with an arbitrary metric space X with a sigma finite measure $\sigma(dx)$.

$\langle F\rangle_\nu = \int_{\mathcal{S}'} F(\phi)\, d\nu(\phi)$.

In this article, we consider the perturbation theory for a "free" probability measure ν_0 on $(\mathcal{S}', \mathcal{B})$ which is subject to the following conditions

1. ν_0 is supported on continuous functions;
2. All moments of ν_0 exist;
3. ν_0 is translation invariant;
4. The translations are mixing [2] on $L^2(\nu_0)$.

The first condition does not hold true for many examples, e.g. the Euclidean free field measures of QFT. In such cases, we tacitly understand the measure ν_0 as the ultra-violet regularized version of the measure of interest. Problems of renormalization would arise in the perturbation series when removing this cut-off. This problem is well-studied in EQFT, where ν_0 is Gaussian. An investigation of renormalization in the general, not necessarily Gaussian, case would be of interest but is beyond the scope of this work, see however Sections 5 and 8 for some first steps. Property no. 2 is an obvious prerequisite for doing perturbation theory w.r.t. polynomial interactions. The remaining properties 3. and 4. technically only become important when discussing thermodynamic limits (removing IR-cut-offs). But they are the main justification for our graphical approach in the next section and that is why we adopt them from the very beginning.

Let $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p\, \phi^p$ be a polynomial with $\bar{p}$ even and $\lambda_{\bar{p}} > 0$, Λ a bounded measurable set in $\mathbb{R}^d$ and ϕ a function from the support of ν_0. We define

$$V_\Lambda(\phi) = \int_\Lambda v(\phi)\, dy\,. \tag{1}$$

For $\phi \in \mathcal{S}' \setminus \mathrm{supp}\nu_0$ we set $V_\Lambda(\phi) = |\Lambda|\, v(0)$ with $|\Lambda|$ the Lebesgue volume of Λ.

Lemma 2.1. *$V_\Lambda : \mathcal{S}' \to \mathbb{R}$ is measurable.*

Proof. Note that $\mathrm{supp}\nu_0$ by definition is a measurable set. For $y \in \Lambda$ and $\phi \in \mathcal{S}'$ define a map $e_y : \mathcal{S}' \to \mathbb{R}$ by setting $e_y(\phi) = 1_{\mathrm{supp}\nu_0}(\phi)\phi(y)$. Then e_y is measurable as a pointwise limit limit of the measurable expressions $1_{\mathrm{supp}\nu_0}(\phi)\langle \delta_y^\epsilon, \phi\rangle$ where δ_y^ϵ is an approximation of the Dirac measure in y by $C_0^\infty(\mathbb{R}^d)$ test functions. Here we needed Condition 1 to establish pointwise convergence. Now, $v(e_y(\phi))$ is measurable in ϕ and continuous in y. The integral (1) thus converges as a Riemannian sum and hence V_Λ is measurable as pointwise limit of measurable functions. ∎

Later on we will feel free to replace the constants λ_p with continuous functions $\lambda_p(y)$, this obviously does not affect Lemma 2.1.

[2] The only invariant functions in that space are in the equivalence class of the multiples of the identity function. Furthermore $\lim_{t\to\infty}\langle F\, H_{ta}\rangle_{\nu_0} = \langle F\rangle_{\nu_0}\langle H\rangle_{\nu_0}$ for $F, H \in L^2(\nu_0)$ $a \in \mathbb{R}^d\backslash\{0\}$ and $H_a(\phi) = H(\phi_a)$ with ϕ_a being the translation of $\phi \in \mathcal{S}'$ by a.

The interacting measure ν_Λ is defined by

$$d\nu_\Lambda(\phi) = Z_\Lambda^{-1}\, e^{-V_\Lambda(\phi)}\, d\nu_0(\phi)\,, \qquad Z_\Lambda = Z(\Lambda, \lambda_0, \ldots, \lambda_{\bar{p}}) = \left\langle e^{-V_\Lambda} \right\rangle_{\nu_0}. \tag{2}$$

In this work, we perturbatively solve the following problems

1. Calculate the moments $Z_\Lambda \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\nu_\Lambda}$ of the non normalized measure $Z_\Lambda \nu_\Lambda$. In particular, for $n = 0$, we calculate the sum over states Z_Λ;
2. Calculate the free energy density $f_\Lambda = \log Z_\Lambda / |\Lambda|$.
3. Calculate the moments $\langle \phi(x_1) \cdots \phi(x_n) \rangle_{\nu_\Lambda}$ of the interacting measure ν_Λ;
4. Remove the infra-red cut-off Λ for the free energy density and the moments of ν_Λ.

The term perturbatively means that we first expand into powers of V_Λ. Take e.g. problem no. 1:

$$\begin{aligned} Z_\Lambda \left\langle \phi(x_1) \cdots \phi(x_n) \right\rangle_{\nu_\Lambda} &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left\langle \phi(x_1) \cdots \phi(x_n) V_\Lambda^m \right\rangle_{\nu_0} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{p_1,\ldots,p_m=0}^{\bar{p}} \int_{\Lambda^m} \lambda_{p_1} \cdots \lambda_{p_m} \langle \phi(x_1) \cdots \phi(x_n) \\ &\qquad \times \quad \phi^{p_1}(y_1) \cdots \phi^{p_m}(y_m) \rangle_{\nu_0}\, dy_1 \cdots dy_m \end{aligned} \tag{3}$$

The first identity in (3) has to be understood in the sense of formal power series in the coupling parameters $\lambda_1, \ldots, \lambda_p$. For many measures of interest, the right hand side of (3) does not converge but (for $\Lambda \subseteq \mathbb{R}^d$ fixed) only gives an asymptotic series, cf. the Lemma 2.2 below. The second identity is due to Fubini's lemma making use of conditions[3] 1) and 2) on ν_0.

Lemma 2.2. *Let $X, V \in \cap_{q\geq 1} L^q(\nu_0)$ with V bounded from below. Then $\langle X e^{-\lambda V} \rangle_{\nu_0}$ at $\lambda = 0$ is infinitely differentiable from the right. Hence, the Taylor series expansion exists at $\lambda = 0$ (but is not necessarily analytic at that point).*

Proof. As $\langle X\, e^{-\lambda V} \rangle_{\nu_0}$ is right differential at $\lambda = 0$ if and only if $e^{-\lambda c} \langle X e^{-\lambda V} \rangle_{\nu_0} = \langle X e^{-\lambda (V+c)} \rangle_{\nu_0}$ is differentiable from the right, we can assume V to be nonnegative. Then $|(e^{-\lambda V} - 1)/\lambda| \leq V$ for $\lambda > 0$ and the differential quotient can be done inside the expectation bracket by Lebesgue theorem. For $\lambda \geq 0$ the right derivative is $\langle XV\, e^{-\lambda V} \rangle_{\nu_0}$ and now the argument can be iterated as $XV \in \cap_{q\geq 1} L^q(\nu_0)$. ∎

To evaluate the perturbation series, one has to calculate the m-th summand on the right hand side of Eq. (3). It obviously only depends on the moments of the free measure ν_0. One can argue that for a ergodic measure the truncated moment functions (to be

[3]The technical formulation of condition 1) and 2) should include that $\phi(x_1) \cdots \phi(x_n)$ are $L^1(\nu_0)$-integrable for all $x_1, \ldots, x_n \in \mathbb{R}^d$ and that the moments of ν_0 are continuous in $x_1, \ldots, x_n$.

defined below) are more "elementary" than the moments themselves and there are interesting examples that illustrate this point of view. It is therefore desirable, to expand (3) into such "elementary" objects. The combinatorial book-keeping of this expansion will be done utilizing a generalized kind of Feynman graphs.

3 A graphical representation of the combinatorics of truncation

The calculus of generalized Feynman graphs that is being proposed here is a device to decompose the moments in the perturbation series

$$\sum_{p_1,\dots,p_m=0}^{\bar{p}} \int_{\Lambda^m} \lambda_{p_1}\cdots\lambda_{p_m}\langle\phi(x_1)\cdots\phi(x_n)\phi^{p_1}(y_1)\cdots\phi^{p_m}(y_m)\rangle_{\nu_0}\,dy_1\cdots dy_m \tag{4}$$

into truncated[4] objects. In order to explain this point of view, let us recall some well-known facts. For a measure ν_0 that is mixing, we have the cluster property for moments

$$\lim_{t\to\infty}\langle\phi(x_1)\cdots\phi(x_j)\phi(x_{j+1}+at)\cdots\phi(x_n+at)\rangle_{\nu_0} = \langle\phi(x_1)\cdots\phi(x_j)\rangle_{\nu_0}\langle\phi(x_{j+1})\cdots\phi(x_n)\rangle_{\nu_0} \tag{5}$$

and we note that this equation formally is non-linear in ν_0. Passing from ordinary moment functions to truncated (connected) moment functions just provides a linearization of this equation. As objects fulfilling a linear equation often are more simple than objects that fulfill nonlinear constraints, it is a reasonable step to decompose (4) into such truncated objects. Of course, these general considerations have to prove useful when dealing with concrete examples.

Let us now pass on to the technicalities. Let $J \subseteq \mathbb{N}$ be a finite set. The collection of all partitions of J is denoted by $\mathcal{P}(J)$. A partition is a decomposition of J into disjoint, nonempty subsets, i.e. $I \in \mathcal{P}(J) \Leftrightarrow \exists k \in \mathbb{N},\ I = \{I_1,\dots,I_k\},\ I_j \subseteq S,\ I_j \cap I_l = \emptyset\ \forall\ 1 \le j < l \le k,\ \cup_{l=1}^{k} I_l = J$.

Definition 3.1. Let $J \subseteq \mathbb{N}$ be a finite set and $\langle J\rangle_{\nu_0} = \langle\prod_{j\in J}\phi(x_j)\rangle_{\nu_0}$ be the collection of moment functions of ν_0. The truncated moment functions $\langle J\rangle_{\nu_0}^T = \langle\prod_{j\in J}\phi(x_j)\rangle_{\nu_0}^T$ of ν_0 are recursively defined (in $\sharp J \in \mathbb{N}$) as follows:

$$\langle J\rangle_{\nu_0} = \sum_{\substack{I\in\mathcal{P}(J)\\ I=\{I_1,\dots,I_k\}}} \prod_{l=1}^{k} \langle I_l\rangle_{\nu_0}^T \tag{6}$$

[4]Depending on the background, truncated moments are also called "cummulants", "Ursell functions" or "connected Greens functions". The notion "truncated moment functions" or equivalently "truncated Schwinger functions" stems from quantum field theory, which here is the main source of inspiration. In the literature, the term "truncated Greens function" often is used for a evaluation of a graphic object with "amputatded" outer legs. Such objects in this text shall be called "amputaded" (truncated) moment functions.

Also, we sometimes identify $J \subseteq \mathbb{N}$ with the random variable $\prod_{j\in J}\phi(x_j)$. It is well known that

F1. The truncated moment functions are symmetric under permutation of their arguments;

F2. (5) $\Leftrightarrow \lim_{t\to\infty}\langle\phi(x_1)\cdots\phi(x_j)\phi(x_{j+1}+at)\cdots\phi(x_n+at)\rangle^T_{\nu_0} = 0\ \forall n, j \in \mathbb{N}$.

Hence, by F2), truncation in fact "linearizes" (5).

Obviously now one can expand the moment $\langle\phi(x_1)\cdots\phi(x_n)\phi^{p_1}(y_1)\cdots\phi^{p_m}(y_m)\rangle_{\nu_0}$ in (4) into truncated objects. To illustrate, how this allows the passage to generalized Feynman graphs, let us consider a two point function in second order ϕ^4-perturbation theory, i.e. take in (4) $n = m = 2$ and $p_1 = p_2 = 4$ and expand into truncated objects. If we consider one partition, see e.g. the one in Fig. 1, we obtain a graph as follows: We replace all sets in the partition, symbolized in Fig. 1 by ⊂⊃, with a new type of vertex " ∘ " that is connected through edges with all points in that set. This is just a more handy symbol for the same thing. One then obtains the graph in Fig. 1.

We now formalize the considerations of the above example. A graph is a geometrical object which consists of vertices, i.e points in $\mathbb{R}^d$, which can be of different types (in our case: inner/outer, full/ empty, cf. Table 1), and non-directed edges, i.e lines connecting exactly two vertices (intersections of lines are ignored). We use the term "leg" for the part of the edge meeting the vertex, see Fig. 2. A special kind of graphs – generalized Feynman graphs – occur can be associated with the expansion of (4):

Definition 3.2. Let $n, m, p_1, \ldots, p_m \in \mathbb{N}_0$ be fixed. A generalized n-point Feynman graph with m interaction vertices of type $p_1, \ldots, p_m$ is a graph with n outer full vertices $\times$, m inner full vertices $\bullet$ with p_j the number of edges connected to the j-th inner full vertex and an arbitrary number of empty inner vertices $\circ$ with an arbitrary number of edges such that each edge is connected with exactly one vertex of full and one vertex of empty type. By definition, full vertices are distinguishable and have distinguishable legs whereas empty vertices are non distinguishable and have non distinguishable legs[5,6], cf. Fig. 2.

	Full	Empty
Inner	$\bullet$	$\circ$
Outer	$\times$	$\boxtimes$

Table 1: Different types of vertices.

[5]More formally: An empty vertex with non-distinguishable legs is a point in $\mathbb{R}^d$. A full vertex with p legs is given by the elements $\{(y,1),\ldots,(y,p)\}$ where $y \in \mathbb{R}^d$ is the point associated to that vertex and (y,j) are the legs, $j = 1,\ldots,p$. Let $\mathcal{M}_1$ be the collection of all empty vertices and all legs of full vertices. Let $\mathcal{M}_2$ be the set of all non ordered pairs of $\mathcal{M}_1$. A graph is a subset of $\mathcal{M}_2$. A generalized Feynman graph is a graph such that each pair in the graph consists of one point (empty vertex) and one leg of a full vertex.

[6]Note that empty outer vertices will not be needed in this work. They are however useful in connection with generalized renormalization group equations where the flow can be expressed in terms of amputated moment functions and graphs, see [11].

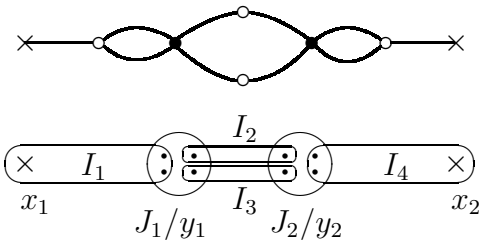


Figure 1: A partition $I = \{I_1, \ldots, I_4\}$ of $n = 2$ outer points x_1, x_2 and $8 = 2 \times 4$ inner points corresponding to one term in the 2nd order perturbation theory of the two point function in ϕ^4 theory. The first "vertex set" J_1 is the set of the first four inner points which take the value y_1 and J_2 the set of the remaining four inner points that take the value the value y_2. Above, the corresponding generalized Feynman graph is displayed.

Let $n, m, p_1, \ldots, p_m \in \mathbb{N}_0$ and $X, J_1, \ldots, J_m \subset \mathbb{N}$ disjoint sets be given s.t. $\sharp X = n$, $\sharp J_1 = p_1, \ldots, \sharp J_m = p_m$. Then we can construct a one to correspondence between $\mathcal{P}(X \cup \bigcup_{l=1}^m J_l)$ and the Feynman graphs with n full outer vertices and m full inner vertices of type $p_1, \ldots, p_m$ that is given in the following way: Pick an arbitrary (but fixed) bijection between the distinguishable outer points and X. Pick also bijections of the legs of the j-th vertex with p_j (distinguishable) edges and J_j, $j = 1, \ldots, m$. Let G be a graph as described in Definition 3.2. Suppose that there are k empty inner vertices in the graph. Give an arbitrary number $l = 1, \ldots, k$ to each inner vertex. For the l-th empty inner vertex let I_l be the set of all points in $X \cup \bigcup_{j=1}^m J_j$ that correspond under the to the given bijections with the edges connected to that empty vertex. The the partition associated to G is given by $I = \{I_1, \ldots, I_k\}$.

Conversely, let $I = \{I_1, \ldots, I_k\} \in \mathcal{P}(X \cup \bigcup_{l=1}^m J_l)$ be given. Draw n outer full vertices, m inner full vertices with $p_1, \ldots, p_m$ legs and k inner empty vertices with $\sharp I_1, \ldots, \sharp I_k$ legs. Connect the legs of l-th inner empty vertex with all the legs of inner full vertices or outer full vertices corresponding – under the fixed bijections – to the points in I_l, $l = 1, \ldots, k$. The result obviously is a generalized Feynman graph. Hence one obtains a mapping from $\mathcal{P}(X \cup \bigcup_{l=1}^m J_l)$ to the generalized Feynman graphs as described in Definition 3.2. The inverse of this mapping clearly is the mapping described in the previous paragraph and vice versa. We have thus deived

Lemma 3.3. *Let $n, m, p_1, \ldots, p_m \in \mathbb{N}_0$ and $J_1, \ldots J_m$ as above. Then there exists a one to one correspondence between $\mathcal{P}(X \cup \bigcup_{l=1}^m J_l)$ and the generalized Feynman graphs as described in Definition 3.2.*

Given the interaction polynomial $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \phi^p$, let $\mathcal{F}(n, m) = \mathcal{F}(n, m, v)$ be the collection of all generalized Feynman graphs with n outer full vertices and m inner full vertices such that each inner full vertex has a number p of edges such that $1 \leq p \leq \bar{p}$ and $\lambda_p \neq 0$. The following definition that assigns a numerical value to each Feynman graph in the physical literature goes under the name "Feynman rules":

Definition 3.4. Let $G \in \mathcal{F}(n, m)$ and $x_1, \ldots, x_n \in \mathbb{R}^d$ be given. Then the real number $\mathcal{V}_\Lambda[G] = \mathcal{V}_\Lambda[G](x_1, \ldots, x_n)$ is obtained in the following way:

Figure 2: Distinguishable and non-distinguishable legs

1. Assign the values $x_1, \ldots, x_n$ to the outer full vertices of the graph and assign arbitrary values $y_1, \ldots, y_m$ to the inner full vertices;

2. For each inner empty vertex with l legs multiply with a truncated l-point moment function with arguments given by the full vertex points where the l edges connected to that vertex are ending;

3. Multiply with λ_p for each inner full vertex with p legs;

4. Integrate the inner full vertices $y_1, \ldots, y_m$ over Λ (w.r.t. the Lebesgue measure).

The value of $\mathcal{V}_\Lambda[G](x_1, \ldots, x_n)$ is obviously just the one of the term in the expansion of (4) into truncated objects that corresponds to the partition associated with G. Note that by F1) this value is independent of the bijections between legs of full vertices and the sets $J_1, \ldots, J_m$. In general, it does depend on the chosen bijection between X and the outer full vertices, this dependence however is eliminated in sums over all generalized Feynman graphs. Combining Lemma 3.3, Definition 3.4 and Equation 4 one thus gets

Theorem 3.5. *The n-point functions of the non-normalized interacting measure $Z_\Lambda \nu_\Lambda$ are given in the sense of formal power series by a sum over all generalized Feynman graphs with n exterior full points that are evaluated according to the Feynman ruled fixed in Def. 3.4, i.e.*

$$Z_\Lambda \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\nu_\Lambda} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{G \in \mathcal{F}(n,m)} \mathcal{V}_\Lambda[G](x_1, \ldots, x_n)\,. \tag{7}$$

All graphs that differ only by the labelling of full inner vertices and edges of inner full vertices give the same value $\mathcal{V}_\Lambda[G]$. The equivalence class of graphs under permutations of legs of full vertices and full vertices is called topological generalized Feynman graph and the perturbation series in Equation (7) can equally be expressed through a sum over topological Feynman graphs where the multiplicity factor, i.e. the number of elements in the equivalence class, is built in into the Feynman rules. Calculating the multiplicity in concrete cases can be rather complicated. A first step in that direction is to make the legs at a full interaction vertex non-distinguishable:

Corollary 3.6. *If one replaces the interaction density $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \, \phi^p$ with $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \, \frac{\phi^p}{p!}$, the generalized Feynman graphs and rules change in the following way : the* •

vertices are treated like vertices with non-distinguishable legs. When evaluating, for each inner full vertex "•" one has to multiply by

$$\frac{1}{\prod_{\substack{\text{all "}\circ\text{" vertices}\\ \text{directly connected}\\ \text{to } \bullet \text{ by an edge}} \sharp\{\text{edges from } \bullet \text{ to } \circ\}!}}. \tag{8}$$

The advantage of this prescription is that treating the edges at the interaction vertex as indistinguishable considerably reduces the combinatorics of generalized Feynman graphs.

A further reduction of this combinatorics takes place, if certain truncated moment functions of ν_0 vanish identically. Then, one can omit the corresponding empty vertices from the perturbation series. A particularly interesting case arises from the following well-known fact

F3. All odd truncated moment functions vanish if and only if all odd moment function vanish.

Corollary 3.7. *Let the measure ν_0 be symmetric under the mapping $\phi \to -\phi$, i.e. $\nu_0(A) = \nu_0(-A)$, $\forall\ A \in \mathcal{B}$. Then one can omit all such generalized Feynman graphs from the perturbation series that have an empty inner vertex with an odd number of legs.*

4 Application to certain functional measures of Lévy type

In this section, we give a justification to the general procedure of Section 3 by the means of examples. In particular, we consider the case where ν_0 is a convoluted generalized white noise measure in the sense of [1]. This gives a unified treatment of the perturbation expansion around the Gaussian Euclidean free field measure in QFT and the case the high temperature expansion of classical, continuous particles in the grand canonical ensemble, cf. [4, 5] and Section 8. In the first – Gaussian – case, generalized Feynman graphs and rules reduce to the classical Feynman graphs and rules. In the more general Lévy case, one still obtains Feynman rules that are very close to the original ones of R. P. Feynman [10, 8]. This simple observation, namely that full and empty vertices in the Feynman rules can be treated on the same level, is the crucial argument in favor of the generalized Feynman graph formalism of Section 3.

Firstly, let us recall some well-known technicalities: Let $\mathcal{C} : \mathcal{S} \to \mathbb{C}$ with $\mathcal{S}$ the space of Schwartz test functionsover $\mathbb{R}^d$. The Frechet derivative of $\mathcal{C}$ at $h \in \mathcal{S}$ in direction $u \in \mathcal{S}$ is by definition $\frac{\partial \mathcal{C}(h)}{\partial u} = \lim_{t\to 0, t\neq 0}(\mathcal{C}(h+tu) - \mathcal{C}(h))/t$ provided this limit exists. The functional derivative of $\mathcal{C}(h)$ w.r.t. $\phi(x)$ is defined as $\frac{\delta \mathcal{C}(h)}{\delta \phi(x)} = \lim_{u\to\delta_x} \frac{\partial \mathcal{C}(h)}{\partial u}$ where δ_x is the Dirac measure of mass one in x an the convergence $u \to \delta_x$ is in the sense of the weak topology of signed Borel measures in $\mathbb{R}^d$.

It is easy to show that the characteristic function $\mathcal{C}_{\nu_0}(h) = \langle e^{i\langle h,.\rangle}\rangle_{\nu_0}$ of the measure ν_0 under the conditions 1. and 2. of Section 2 has functional derivatives of arbitrary order. Obviously, $\mathcal{C}_{\nu_0}$ is the generating functional of the sequence of moments of ν_0, i.e. $\langle\phi(x_1)\cdots\phi(x_n)\rangle_{\nu_0} = (-i)^n \frac{\delta^n \mathcal{C}_{\nu_0}(h)}{\delta\phi(x_1)\cdots\delta\phi(x_n)}|_{h=0}$. Let $\mathcal{C}^T_{\nu_0} = \log \mathcal{C}_{\nu_0}$, then $\mathcal{C}^T_{\nu_0}(h)$ is well defined for $h \in \mathcal{S}$ sufficiently small as $\mathcal{C}_{\nu_0}(h)$ is continuous in h and $\mathcal{C}(0) = 1$. Furthermore, also $\mathcal{C}^T_{\nu_0}$ has functional derivatives of arbitrary order. The crucial fact needed in this section is the basic linked cluster theorem

F4. $\mathcal{C}^T_{\nu_0}$ is the generating functional of the sequence of truncated moment functions.

Minlos theorem [15] establishes a one to one correspondence between characteristic functionals $\mathcal{C} : \mathcal{S} \to \mathbb{C}$ (positive definite normalized $\mathcal{C}(0) = 1$ and continuous) random fields η indexed by $\mathcal{S}$ (up to equivalence in law), cf. [12], and probability measures ρ_0 on $(\mathcal{S}', \mathcal{B})$ given by $\mathcal{C}(h) = \langle e^{\langle h,.\rangle}\rangle_{\rho_0} = \mathbb{E}[e^{i\eta(h)}]$. To define a measure ρ_0, it is thus sufficient to write down its characteristic functional. Let us do this for noise (infinitely divisible and non-correlated at a distance) measures of Lévy type.

Let $\psi : \mathbb{R} \to \mathbb{C}$ be a Lévy characteristic (conditionally positive definite, normalized $\psi(0) = 0$ and continuous) [7] that is infinitely often differentiable at zero. Then, $\psi(t)$ has the following representation

$$\psi(t) = iat - \frac{\sigma^2}{2}t^2 + z\int_{\mathbb{R}\setminus\{0\}} (e^{ist} - 1)\, dr(s), \tag{9}$$

where $a \in \mathbb{R}$, $\sigma^2, z \geq 0$ and r is a probability measure on $\mathbb{R} \setminus \{0\}$ that has all moments. The first term in (9) is called deterministic, the second one Gaussian part and the third one Poisson part. If $z > 0$, the representation (9) is unique. It is well-known, cf. Theorem 6 of [9] p. 238, that $\mathcal{C}_{\rho_0}(h) = \exp\{\int_{\mathbb{R}^d} \psi(h)\, dx\}$, $h \in \mathcal{S}$, defines a characteristic functional.

Let ρ_0 be the associated measure on $(\mathcal{S}', \mathcal{B})$ and η the associated coordinate process, i.e. $\eta(h)(\omega) = \omega(h)\ \forall h \in \mathcal{S}, \omega \in \mathcal{S}'$. We consider the linear stochastic partial differential equation (SPDE) $L\phi = \eta$ with $L : \mathcal{S}' \to \mathcal{S}'$ a partial (pseudo) differential operator with constant coefficients and with Greens function $g : \mathbb{R}^d \to \mathbb{R}$, i.e. $g * L\omega = \omega$ for $\omega \in \mathcal{S}'$. As the most relevant case, we consider $L = (-\Delta + m_0^2)^\alpha$ for $\alpha > 0, m_0 > 0$ and Δ the Laplacian on $\mathbb{R}^d$. Then, the solution to this SPDE $\phi = g * \eta$ exists pathwisely. As a canonical process it is equivalent (in distribution) to the coordinate process of the measure ν_0 on $(\mathcal{S}', \mathcal{B})$ with characteristic functional

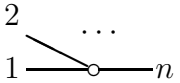
$$\mathcal{C}_{\nu_0}(h) = \exp\{\int_{\mathbb{R}^d} \psi(g * h)\, dx\}. \tag{10}$$

It is easily verified that for $L = (-\Delta + m_0^2)^{\frac{1}{2}}$ and $a, z = 0$, ν_0 is the free field measure of Euclidean QFT (Nelson's free field measure, cf. [10, 23]). But also in the more general case considered here, connections with quantum field theory can be made explicit [1].

It turns out [1] that the measure ν_0 obtained in this way fulfills the conditions 2. – 3. of Section 2, however in general does not fulfill Condition 1. This can be seen as a ultra-violet problem and can be removed replacing g with $g_\epsilon = g * \chi_\epsilon$ where $\chi_\epsilon \in \mathcal{S}$ is an

approximation of the Dirac delta distribution in zero, $\chi_\epsilon \to \delta_0$ in $\mathcal{S}'$. The measure ν_0^ϵ with characteristic functional (10) where g is replaced by g_ϵ then also fulfills Condition 1[7]. In the following we will tacitly assume that measures ν_0 are suitably ultra-violet regularized and we do not write the superscript ϵ. Some simple examples, where the ultra-violet cut-off can be removed in the perturbation series can be found in the Sections 5 and 8. The ultra-violet problem for the general case of convoluted Lévy noise has to be postponed.

Combination of F4. with (10) now yields



$$= \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\nu_0}^T = c_n \int_{\mathbb{R}^d} g(x_1 - z) \cdots g(x_n - z)\, dz\,, \tag{11}$$

where

$$c_n = (-i)^n \frac{d^n \psi(t)}{dt^n}|_{t=0} = \delta_{n,1}\, a + \delta_{n,2}\, \sigma^2 + z \int_{\mathbb{R}\setminus\{0\}} s^n\, dr(s)\,, \tag{12}$$

$\delta_{n,n'}$ being the Kronecker symbol. Note that the property F2. obviously holds for the truncated moments (11) for g of sufficiently fast decay. From equation (11) one now obtains the Feynman rules for convoluted Lévy type noise:

Theorem 4.1. *Let $G \in \mathcal{F}(n,m)$ and $x_1, \ldots, x_n \in \mathbb{R}^d$ be given. For the case of a convoluted Lévy noise measure ν_0 the value of $\mathcal{V}_\Lambda[G] = \mathcal{V}_\Lambda[G](x_1, \ldots, x_n)$ can be calculated as follows:*

1. *Assign the values $x_1, \ldots, x_n$ to the outer full vertices of the graph and assign arbitrary values $y_1, \ldots, y_m$ to the inner full vertices and $z_1, \ldots, z_k$ to the inner empty vertices where k is the number of such vertices;*

2. *For each edge in the graph going from a full vertex x_j or y_j to an empty vertex z_q multiply with the "propagator function" $g(x_j - z_q)$ and $g(y_j - z_q)$, respectively;*

3. *For each inner empty vertex with l legs multiply with c_l;*

4. *Multiply with λ_p for each inner full vertex with p legs;*

5. *Integrate over all inner vertices $y_1, \ldots, y_m$ and $z_1, \ldots, z_k$ (w.r.t. the Lebesgue measure) – full vertices are being integrated over Λ and empty ones over $\mathbb{R}^d$.*

In Theorem 4.1, the constants c_l, $l \in \mathbb{N}$, take the rôle of coupling constants of empty vertices. Hence empty and full inner vertices in the Feynman rules are treated on the same level – at least in the thermodynamic limit $\Lambda \nearrow \mathbb{R}^d$.

Let us consider the centered Gaussian case $a = z = 0$ as a special case. Then, $c_l = 0$ for $l = 1$ and $l \geq 3$, cf. (10). Hence all graphs containing empty vertices with a number of legs not equal to two give a zero contribution. The remaining two legged empty vertices $1 \multimap 2 = c_2 g * g(x_1 - x_2)$ can be identified with a straight line of a new type. Hence one obtains the classical Feynman graphs and -rules as a special case:

[7] ν_0^ϵ is the image measure of ν_0 under the mapping $\mathcal{S}' \ni \phi \to \phi * \chi_\epsilon \in C^\infty(\mathbb{R}^d)$.

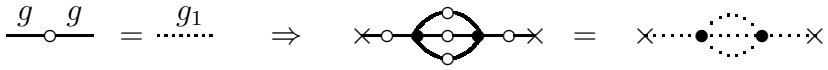


Figure 3: Identification of a generalized Feynman graph with propagator g with a classical Feynman graph with propagator g_1 in the case of a Gaussian functional measure.

Corollary 4.2. *In the case where ν_0 is a centered Gaussian measure, i.e. $a = z = 0$, there exists a one to one correspondence between the generalized Feynman graphs that give non-zero contributions in the Feynman rules, i.e. that contain only two-legged empty vertices, and the classical Feynman graphs, cf. Fig. 3.*

Furthermore, the generalized Feynman rules of Theorem 4.1 with propagator function g applied to a generalized Feynman graph and the classical Feynman rules applied to the corresponding classical Feynman graph with propagator $g_1 = c_2\, g * g$ give the same result.

Remark 4.3. In [1] the moment functions of convoluted Lévy noise have been analytically continued to vacuum expectation values (Wightman functions) of a local, relativistic QFT that fulfill all Wightman axioms [26] except positivity. Thus, for the non-interacting case, there is a correspondence between convoluted Lévy noise and a relativistic, local quantum field theory with indefinite metric [2].

It is an interesting speculation that this correspondence exists also in the interacting case. We note that by Theorem 4.1 all Feynman graphs correspond to a Feynman graph in some Gaussian theory with modified propagator and interaction structure. The contribution to the Wightman function that corresponds to such a graph, is known at least in principle, i.e. in non-renormalized form [16, 24, 25]. It is natural to conjecture, that the analytic continuation of the function corresponding to the Euclidean Feynman graph is given by that part of the Wightman function. One can show that the analytic continuation obtained in [1] is equal to the expression in [16, 24, 25] obtained for the sectorized star graph. Hence this conjecture holds for star graphs (11).

If it would be true in general, one would obtain a perturbative correspondence of convoluted Lévy noise with local, polynomial interactions and local, relativistic Quantum fields with indefinite metric. In particular, the expectation values of products of the static field of a ensemble of interacting particles, see [4, 5] and Section 8 for further explanations, would have a relativistic, local Wightman function as its counterpart, as already conjectured in [5]. The above argument above gives new evidence in favor of this conjecture. ■

5 Wick–ordering vs Wiener-Itô-Segal chaos decomposition

In this section we give the notion of Wick ordering for a general functional measure. When removing ultra-violet cut-offs, take e.g. $\epsilon \searrow 0$ in the examples given in Section 4,

Figure 4: Self-contractions at a ϕ^4 interaction vertex.

some graphs in the expansion introduced in Section 3 will diverge. The reason for these divergences is that the truncated moment functions of a non-uv-regularized measure have singularities when two or more of its arguments coincide. The worst of these cases, i.e. the one with the strongest divergences, certainly is the one when all of the arguments of a truncated n-point moment function coincide and a term $\sim \langle \phi^n(y) \rangle_{\nu_0}^T$ occurs in the Feynman rules. In graphical terms, this situation corresponds to a self-contraction, i.e. to the case where all n legs of an empty vertex are connected to one and the same inner full vertex, cf. Fig. 4. Wick ordering – as it is understood here – removes graphs with self-contractions from the perturbation series. In the general case, this does not yet render the perturbation series finite, and more sophisticated procedures of renormalizataion have to be applied to achieve that. Some remarkable exceptions – Gaussian and not – in $d = 2$ dimensions will be discussed at the end of this section. We also clarify the relation of Wick ordering in the given sense and the decomposition of $L^2(\nu_0)$ by means of orthogonal polynomials, e.g. of Hermite [10, 23] or Charlier [13, 17] type that goes under the name of Wiener-Itô-Segal chaos decomposition.

Let $X \subseteq \mathbb{N}$ a set of numbers and $Y \in L^2(\nu_0)$ a random variable. When considering $X \cup \{Y\}$ as a collection of $\sharp X + 1$ objects, we can use Definition 3.1 to make sense of $\langle X\, Y \rangle_{\nu_0}^{(T)} = \langle \prod_{j \in X} \phi(x_j) Y \rangle_{\nu_0}^{(T)}$. Here the symbol (T) is being used instead of T in order to symbolize that the random variable Y in the combinatorics of Def. 3.1 is treated as one object in order to avoid ambiguities if e.g. $Y = \phi(y_1) \cdots \phi(y_n)$. The field entries from $X = \prod_{j \in X} \phi(x_j)$ combinatorially are treated as distinct objects. We are now looking for another random variable, denoted by $: X :=: X :_{\nu_0}$, that has the same $L^2(\nu_0)$ inner product with an arbitrary $L^2(\nu_0)$ random variable as X with the exception that there are no self-contractions in X, i.e.

$$\langle : X : Y \rangle_{\nu_0} = \langle X\, Y \rangle_{\nu_0}^{(T)} \quad \forall Y \in L^2(\nu_0). \tag{13}$$

By Def. 3.1 one has

$$\begin{aligned} \langle X\, Y \rangle_{\nu_0} &= \sum_{\substack{I \in \mathcal{P}(X \cup \{Y\}) \\ I = \{I_1, \ldots, I_k\}}} \prod_{l=1}^{k} \langle I_k \rangle_{\nu_0}^{(T)} \\ &= \sum_{\substack{I \in \mathcal{P}(X) \\ I = \{I_1, \ldots, I_k\}}} \sum_{j=1}^{k} \langle I_j Y \rangle_{\nu_0}^{(T)} \prod_{\substack{l=1 \\ l \neq j}}^{k} \langle I_l \rangle_{\nu_0}^{T} + \langle Y \rangle_{\nu_0} \sum_{\substack{I \in \mathcal{P}(X) \\ I = \{I_1, \ldots I_k\}}} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^{T} \,, \end{aligned} \tag{14}$$

We note that only the $k = 1$ term in the first sum of (14) appears on the right hand side

of (13). Hence, using linearity in Y, one can show that the following is the only solution to (13):

Definition 5.1. For $X \subseteq \mathbb{N}$ with $\sharp X = 1$ let $: X := X - \langle X \rangle_{\nu_0}$. Let now $X \subseteq \mathbb{N}$ with $\sharp X > 1$ and suppose that $: J :$ is already defined for $J \subseteq \mathbb{N}$ with $\sharp J < \sharp X$. Then

$$: X := X - \langle X \rangle_{\nu_0} - \sum_{\substack{I \in \mathcal{P}(X) \\ I = \{I_1, \ldots, I_k\},\, k>1}} \sum_{j=1}^{k} : I_j : \prod_{\substack{l=1 \\ l \neq j}}^{k} \langle I_l \rangle_{\nu_0}^T . \tag{15}$$

recursively defines[8] the Wick ordered monomial $: X :=: \prod_{j \in X} \phi(x_j) :_{\nu_0}$.

It remains to show that this definition also solves the problem of removing the self-contractions from the perturbation series. Let $J_1, \ldots J_m, X \subseteq \mathbb{N}$ be disjoint finite sets. A partition $I \in \mathcal{P}(\cup_{l=1}^m J_l \cup X)$, $I = \{I_1, \ldots, I_k\}$, by definition has a self-contraction at a set J_l, $l \in \{1, \ldots, m\}$, if $\exists j \in \mathbb{N}$, $1 \leq j \leq k$, such that $I_j \subseteq J_l$. The collection of all partitions I that do not have self-contractions at J_l for $l = 1, \ldots, m$ is denoted by $\mathcal{P}^{\mathrm{Wick}}(J_1, \ldots, J_m; X)$.

Proposition 5.2. *Let $J_1, \ldots J_m$ and X as above. Then*

$$\langle : J_1 : \cdots : J_m : X \rangle_{\nu_0} = \sum_{\substack{I \in \mathcal{P}^{\mathrm{Wick}}(J_1, \ldots, J_m; X) \\ I = \{I_1, \ldots I_k\}}} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^T . \tag{16}$$

Proof. The proof is by induction over $q = \sum_{l=1}^m \sharp J_l$. $q = 0$ is just Definition 3.1. Suppose that (16) holds up to $q - 1$. Then, by definition of Wick ordering,

$$\begin{aligned} \langle : J_1 : \cdots : J_m : X \rangle_{\nu_0} &= \langle : J_1 : \cdots : J_{m-1} : J_m \, X \rangle_{\nu_0} - \langle J_m \rangle_{\nu_0} \langle : J_1 : \cdots : J_{m-1} : X \rangle_{\nu_0} \\ &- \sum_{\substack{Q \in \mathcal{P}(J_m) \\ Q = \{Q_1, \ldots, Q_k\},\, k>1}} \sum_{j=1}^{k} \langle : J_1 : \cdots : J_{m-1} :: Q_j : X \rangle_{\nu_0} \prod_{\substack{l=1 \\ l \neq j}}^{k} \langle Q_l \rangle_{\nu_0}^T . \end{aligned} \tag{17}$$

Application of the induction hypothesis to the right hand side yields

$$\begin{aligned} &\sum_{\substack{I \in \mathcal{P}^{\mathrm{Wick}}(J_1, \ldots, J_{m-1}; J_m \cup X) \\ I = \{I_1, \ldots I_k\}}} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^T - \sum_{\substack{Q \in \mathcal{P}(J_m) \\ Q = \{Q_1, \ldots Q_k\}}} \sum_{\substack{P \in \mathcal{P}^{\mathrm{Wick}}(J_1, \ldots, J_{m-1}; X) \\ P = \{P_1, \ldots, P_{k'}\}}} \prod_{l=1}^{k} \langle Q_l \rangle_{\nu_0}^T \prod_{l'=1}^{k'} \langle P_{l'} \rangle_{\nu_0}^T \\ &- \sum_{\substack{Q \in \mathcal{P}(J_m) \\ Q = \{Q_1, \ldots, Q_k\},\, k>1}} \sum_{j=1}^{k} \sum_{\substack{P \in \mathcal{P}^{\mathrm{Wick}}(J_1, \ldots, J_{m-1}, Q_j; X) \\ P = \{P_1, \ldots P_{k'}\}}} \prod_{l'=1}^{k'} \langle P_{l'} \rangle_{\nu_0}^T \prod_{\substack{l=1 \\ l \neq j}}^{k} \langle Q_l \rangle_{\nu_0}^T . \end{aligned} \tag{18}$$

[8] Note that for $I = \{I_1, \ldots I_k\}$ with $k > 1$, $\sharp I_j < \sharp X$ for $j = 1, \ldots, k$.

In the first sum we find all partitions I of $\cup_{l=1}^{m} J_l \cup X$ that do not have self-contractions at $J_1, \dots J_{m-1}$. As in the second sum $I = Q \cup P$ is a partition of the same set, we can identify this sum with the sum over all partitions I that do not have self-contractions at $J_1, \dots, J_{m-1}$ and where all points from J_m are contained in self-contractions. To complete the proof, the third sum finally has to be identified with the sum over all partitions that do not contain a self-contraction at $J_1, \dots, J_{m-1}$ and do contain at least one self-contraction at J_m, however not all points in J_m are being self-contacted.

Let Q, j and P be given from the third sum. Then $I = Q \setminus \{Q_j\} \cup P$ is such a partition: As $Q \setminus \{Q_j\} \neq \emptyset$ there are self-contractions at J_m, however the points in $Q_j \neq \emptyset$ are not contained in a self-contraction.

Let, on the other hand, I be a partition of $\cup_{l=1}^{m} J_l \cup X$ from the set of partitions described above. Firstly, for $Q \in \mathcal{P}(J_m)$, $\sharp Q = k$, we fix an enumeration $1, \dots, k$ of the elements of Q (independently of I). Let $\tilde{Q} = \{Q' \in I : Q' \subseteq J_m\} \neq \emptyset$, $Q^c = J_m \setminus \cup_{Q' \in \tilde{Q}} Q' \neq \emptyset$ and $k = \sharp \tilde{Q} + 1 > 1$. Let $Q = \tilde{Q} \cup \{Q^c\} \in \mathcal{P}(J_m)$ and j be the number of the element Q^c. Furthermore, we set $P = \{P' \cap [\cup_{l=1}^{m-1} J_l \cup Q^c \cup X] : P' \in I\} \setminus \{\emptyset\}$. Then, $P \in \mathcal{P}^{\mathrm{Wick}}(J_1, \dots J_{m-1}, Q^c; X)$ and we get a map from the prescribed set of partitions to the index set of the third sum.

It is easy to check that the two maps between the described set of partitions and the index set of the third sum of (18) (the other way round, respectively) that have been constructed in the preceding two paragraphs are the inverses of each other. Hence the correspondence between the two sets is one to one. Finally, the contribution to the third sum determined by Q, j and P coincides with the contribution associated to the corresponding $I = I(Q, j, P)$. ∎

We can now define the p-th Wick power $: \phi^p : (x) =: \phi(x_1) \cdots \phi(x_n) : |_{x_1, \dots, x_n = x}$. Obviously, $: \phi^p : (x)$ is a polynomial in the random variable $\phi(x)$ with coefficients determined recursively according to Def. 5.1 from the values of $C_n = \langle \phi^n(x) \rangle_{\nu_0}^T$, $n < p$. By properties 1. and 2. of ν_0 (see Section 2), C_n is finite and by property 3. it does not depend on x. Hence, $: \phi^p :=: \phi^p :_{\nu_0}$ is a well-defined polynomial in $\phi \in \mathbb{R}$. We also call this polynomial the p-th Wick power. The main result of this section is:

Theorem 5.3. *Let $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \phi^p$ and $: v(\phi) := \sum_{p=0}^{\bar{p}} \lambda_p : \phi^p :$. If one replaces the interaction polynomial v by its Wick-ordered counterpart $: v :$, the perturbation series given in Theorem 3.5 remains the same with the only exception that all generalized Feynman graphs that contain self-contractions at inner full vertices are removed from the series.*

Proof. Note that a generalized Feynman graph has a self-contraction at an inner full vertex if and only if the corresponding partition (see Section 3) has a self-contraction at the corresponding set of points J_l, $l = 1, \dots, m$ (see also Figs.1 and 4). The theorem thus follows from Proposition 5.2. ∎

For a centered Gaussian measure, Wick ordering of the interaction vertex means that no dashed line (see Fig. 3) leaving the vertex can return to the same vertex, i.e. all Graphs containing a sub-graph  are deleted from the perturbation series. This is of course the well-known graphical meaning of Gaussian Wick-ordering.

Figure 5: Contributions to $\langle : \phi(x_1) \cdots \phi(x_n) :: \phi(y_1) \cdots \phi(y_m) : \rangle_{\nu_0}$ for a) the Gaussian case for $n = m$, $\pi \in \mathrm{Perm}(n)$, and b) the non Gaussian case for $n = 1, m > 1$.

In the centered Gaussian case, Wick ordered monomials $: J_1 :=: \phi(x_1) \cdots \phi(x_n) :$, $: J_2 :=: \phi(y_1) \cdots \phi(y_m) :$ with a different number of points $n \neq m$ are orthogonal in $L^2(\nu_0)$, as it is not possible to make pairings out of $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ without getting at least one self-contraction at $J_1 = \{x_1, \ldots, x_n\}$ or $J_2 = \{y_1, \ldots, y_m\}$. If $n = m$, the only possible contributions are those of Fig. 5 a), and hence $\langle : J_1 : : J_2 : \rangle_{\nu_0} = n! \, \langle \mathrm{Sym} \otimes_{l=1}^n \delta_{x_l}, \mathrm{Sym} \otimes_{l=1}^n \delta_{y_l} \rangle_n$ with $\langle ., . \rangle_n$ being the scalar product on $\mathcal{H}^{\otimes n}$ and $\mathcal{H}$ the one particle Hilbert space given by the closure of $C_0^\infty(\mathbb{R}^d)$ w.r.t. the inner product $\langle u, h \rangle = \int_{\mathbb{R}^{2d}} u(x) h(y) \langle \phi(x) \phi(y) \rangle_{\nu_0} \, dxdy$, $u, h \in C_0^\infty(\mathbb{R}^d)$. Using property 1. from Section 2, it is easy to prove that $\delta_x \in \mathcal{H}$. Sym stands for symmetrization. As the span of $: J :$, $J \subseteq \mathbb{N}$ is dense in $L^2(\nu_0)$, one obtains the Wiener-Itô-Segal isomorphism between $L^2(\nu_0)$ and the Bosonic Fock space over $\mathcal{H}$. For the details we refer to [10, 23].

As a self-contraction $\bullet\!\!-\!\!\circ$ can not occur at a Wick-ordered interaction vertex (or a monomial), the above considerations also hold in the non-centered Gaussian case where $C_1 = -\!\!\circ \neq 0$.

A functional measure ν_0 is non-Gaussian if and only if $\exists n > 2$ such that $\langle \phi(x_1) \cdots \phi(x_n) \rangle_{\nu_0}^T \neq 0$ for some values of $x_1, \ldots, x_n \in \mathbb{R}^d$. Let $m + 1$ be the smallest such number and $\{x_1, y_1, \ldots, y_m\}$ be a collection of points such that the truncated $m+1$-point function does not vanish. Obviously, the $L^2(\nu_0)$ inner product of $: J_1 :$ and $: J_2 :$, $J_1 = \{x_1\}$ and $J_2 = \{y_1, \ldots, y_m\}$, consists out of only one non-zero contribution depicted graphically in Fig. 5 b). For non-Gaussian measures our graphical definition of Wick ordering does not give an orthogonal decomposition of $L^2(\nu_0)$.

Corollary 5.4. *Wick ordering as defined in Def. 5.1 gives an orthogonal decomposition of $L^2(\nu_0)$ in the sense of a Wiener-Itô-Segal isomorphism with the Bosonic Fock space if and only if ν_0 is Gaussian.*

To close this section, let us consider non-Gaussian some examples in $d = 2$ where Wick ordering renders the entire perturbation series finite. If one however considers the non-Gaussian generalization of Nelsons free field given by $L\phi = \eta$ with $L = (-\Delta + m_0^2)^{1/2}$ and η a non-Gaussian noise field, i.e. $z > 0$ in (9), see Section 4, one can easily see from $g(x) \sim 1/|x|$ for small $|x|$ and Theorem 4.1 that Wick ordering does not remove all divergences: Take, e.g. for a $: \phi^4 :$-interaction in $d = 2$ the generalized Feynman graph in Fig. 1 which diverges logarithmically. For the models of Section 4, Wick-ordering in $d = 2$ thus is less efficient than in the Gaussian case.

There is however a modification of these models [3] where Wick ordering in $d = 2$ dimensions removes all divergences. Here we briefly recall the construction. Let η be a

Lévy noise field, cf. Section 4, and let $\tilde{\eta}$ be a Gaussian field with characteristic functional $\mathcal{C}_{\tilde{\eta}}(h) = \exp\{-\frac{c_2}{m_0^2}\int_{\mathbb{R}^d} |\nabla h|^2\, dx\}$, $h \in \mathcal{S}$ with c_2 as in (12). ∇ is the gradient on $\mathbb{R}^d$. $\tilde{\eta}$ still is an infinite divisible, ultralocal field, i.e. a field that has no correlations at a distance. We study the linear SPDE $L\phi = \eta + \tilde{\eta}$ for $L = (-\Delta + m_0^2)$ where η and $\tilde{\eta}$ are assumed to be independent. Let g be the Greens function of L. Then the solution ϕ of this SPDE has characteristic functional

$$\mathcal{C}_{\nu_0}(h) = \exp\{\int_{\mathbb{R}^d} [\psi(g * h) - \frac{c_2}{m_0^2}|\nabla(g * h)|^2]\, dx\}\ ,\ h \in \mathcal{S}. \tag{19}$$

Performing the functional derivatives of $\log \mathcal{C}_{\nu_0}(h)$ at $h = 0$ one obtains that the truncated moment functions for $n \neq 2$ are given by (11). For $n = 2$ one obtains due to the correction term induced by $\tilde{\eta}$: $\langle \phi(x)\phi(y)\rangle^T_{\nu_0} = \tilde{c}_2\, g(x - y)$ with $\tilde{c}_2 = c_2/m_0^2$.

The Feynman rules of Theorem 4.1 now change as follows: Put a propagator $\tilde{c}_2 g(y-y')$ for all subgraphs $y \multimap y'$. Then proceed as in Theorem 4.1 for the remaining vertices and edges[9]. For the given L, the singularity of $g(x)$ at $x = 0$ is only logarithmic $\sim -\frac{1}{2\pi}\log|x|$ and $g(x) \sim e^{-m_0|x|}$ for $|x|$ large. As there are no self-contractions in a Wick ordered gen. Feynman graph and arbitrary powers of g are integrable, one obtains:

Theorem 5.5. *Let $d = 2$ be the dimension of the underlying space, ν_0 constructed as above and $: v(\phi) := \sum_{p=0}^{\bar{p}} \lambda_p : \phi^p :$ the interaction density. Then the perturbation series of ν_0 is free of divergences, i.e. the perturbation series of the ultra-violet regularized measures ν_0^ϵ with interaction densities $: v(\phi) :_\epsilon = \sum_{p=0}^{\bar{p}} \lambda_p : \phi^p :_{\nu_0^\epsilon}$ converges term by term as $\epsilon \searrow 0$.*

Proof. Let G be a generalized Feynman graph. By the Feynman rules described above, all the values $\mathcal{V}_\Lambda[G](x_1, \ldots, x_n)$ (up to constants) occur also in the perturbation theory of some Gaussian $P(\phi)_2$-theory. The proof thus is essentially[10] the same as in [10] Lemma 8.5.2 and Theorem 8.5.3. ∎

We note that the models described in Theorem 5.5 in particular include Nelson's free field, take $z = 0$ in (9). For $z > 0$, the measures ν_0 are non-Gaussian. Even though the truncated moment functions $\langle \phi(x_1) \cdots \phi(x_n)\rangle^T_{\nu_0}$ of such ν_0 are continuous functions for $n \geq 3$ and hence the constants $C_n^\epsilon = \langle \phi^n(x)\rangle^T_{\nu_0^\epsilon}$ are finite in the limit $\epsilon \searrow 0$, one cannot replace the Wick ordering in the perturbation series w.r.t. ν_0 with the Wick-ordering w.r.t. a Gaussian measure with the same covariance functions if one wants to get a finite perturbation series. We take e.g. a $: \phi^p :_{\nu_0^\epsilon}$ interaction for a symmetric measure (cf. Corollary 3.7). For $p = 2$ Gaussian and non-Gaussian Wick ordering coincide. For $p = 4$ they still coincide up to a constant C_4^ϵ that converges for $\epsilon \searrow 0$ and can be neglected.

[9] An alternative description is to draw generalized Feynman graphs with two kinds of edges, dotted and not, and inner empty vertices that have three or more legs (for η centered). Dotted edges go from full to full vertices, non-dotted edges from empty to full vertices. Then the graph can be evaluated as in Theorem 4.1 if one multiplies with an extra $\tilde{c}_2$ for each dotted edge.

[10] Here, infra-red cut-offs have to be treated slightly more carefully as empty vertices do not have such a cut-off. One can take this into account by integrating first over the empty vertices and then over the full ones. Note that every connected component of a generalized Feynman graph contains at least one full vertex that provides an IR-cut-off.

For $p = 6$ however, in the difference there is an additional constant term $\sim C_4^\epsilon C_2^\epsilon$ that diverges, but this can still be considered as an irrelevant ground state energy. Finally, for $p = 8$ there is a logarithmically divergent mass-counterterm $\sim C_4^\epsilon C_2^\epsilon : \phi^2 :_{\nu_0^\epsilon}$ present in the non-Gaussian Wick ordering that is missing in the Gaussian one. This makes it clear that one cannot hope for a finite perturbation series using the wrong (Gaussian) Wick ordering if $p \geq 8$.

6 Linked cluster theorem for generalized Feynman graphs

In this section we solve the Problem 2 and the first part of Problem 4 of Section 2, i.e. we perturbatively calculate the free energy density $f_\Lambda = \log Z_\Lambda/|\Lambda|$ and we prove the existence of the thermodynamic (TD) limit $\Lambda \nearrow \mathbb{R}^d$ for each term in the perturbation series for a general ν_0 with a sufficiently fast clustering, cf. property 4 of Section 2, F2. and Eq. (5). The result is the expected one – only connected generalized Feynman graphs contribute to f_Λ – and can be seen as one of the many variations of the linked cluster theorem. As the method of proof, we do not use polymer systems, see e.g. [6, 21], but use bookkeeping of partitions instead.

First we note that $Z_\Lambda(\Lambda, \beta\lambda_0, \dots \beta\lambda_{\bar{p}}) = \langle e^{-\beta V_\Lambda} \rangle_{\nu_0}$ is the Laplace transform of the random variable V_Λ in the parameter $\beta > 0$. If we want to expand into powers V_Λ, we can expand in powers of β and put $\beta = 1$ afterwards. The this expansion of course is the one obtained in Section 3 for $n = 0$. If we now want to expand the free energy density $f_\Lambda = f(\Lambda, \lambda_0, \dots, \lambda_{\bar{p}}) = \log Z_\Lambda(\Lambda, \lambda_0, \dots, \lambda_{\bar{p}})/|\Lambda|$ into powers of V_Λ, we can do the same for $f(\Lambda, \beta\lambda_0, \dots, \beta\lambda_{\bar{p}})$. By the basic linked cluster theorem F4 in Section 4, see also Appendix A, we get in the sense of formal power series

$$\log Z_\Lambda = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \langle V_\Lambda^m \rangle_{\nu_0}^{(T)} , \tag{20}$$

where the superscript (T) means that in the combinatorics of Def. 3.1 each of the m copies of V_Λ is treated as one object, even if they contain higher powers of the field variables $\phi(x)$. As already in the preceding section, the superscript T is reserved for the combinatorics in Def. 3.1 where each copy of $\phi(x)$ is treated as one object. As the latter combinatorics is linked with generalized Feynman graphs, we have to expand $\langle V_\Lambda \rangle_{\nu_0}^{(T)}$ in terms of truncated moments $\langle \phi(x_1) \cdots \phi(x_n) \rangle_{\nu_0}^T$. The first step is to prove the (T)-truncated analogue of Eq. (3), i.e. that one can interchange the truncated expectation and the integrals over Λ:

Lemma 6.1. *For V_Λ defined as in Eq. (1) the following holds:*

$$\langle V_\Lambda^m \rangle_{\nu_0}^{(T)} = \sum_{p_1, \dots, p_m}^{\bar{p}} \int_{\Lambda^m} \lambda_{p_1} \cdots \lambda_{p_m} \langle \phi^{p_1}(y_1) \cdots \phi^{p_m}(y_m) \rangle_{\nu_0}^{(T)} \, dy_1 \cdots dy_m \, . \tag{21}$$

Here the superscript (T) means that in Def. 3.1 each random variable $\phi^{p_l}(y_l)$, $l = 1, \dots, m$, and each of the m copies of V_Λ is being treated as one object.

Proof. By Fubini's theorem (21) is true if we omit the (T) on both sides. For $m = 1$ the $\langle V_\Lambda \rangle_{\nu_0}^{(T)} = \langle V_\Lambda \rangle_{\nu_0}$ and $\langle \phi^{p_1}(y_1) \rangle_{\nu_0}^{(T)} = \langle \phi^{p_1}(y_1) \rangle_{\nu_0}$. Hence (21) also holds for $m = 1$. For $m > 1$ we get by induction and Def. 3.1 that the difference between the left hand side and the right hand side of (21) without (T) consists only out of the truncated terms on both sides with the partition $I = \{\{1, \ldots, m\}\}$ and hence out of the difference of both sides of (21) with the superscript (T). This difference must thus be zero. ∎

Let $J_1, \ldots, J_m \subseteq \mathbb{N}$ be disjoint sets. By definition, a partition $I \in \mathcal{P}(\cup_{l=1}^m J_l)$ is connected w.r.t. the "blocks" $J_1, \ldots, J_m$, in notation $I \in \mathcal{P}_c(J_1, \ldots, J_m)$, if for $I = \{I_1, \ldots, I_k\}$ $\nexists\, 1 \leq i_1, \ldots, i_q \leq k$, $1 \leq q < k$ and $1 \leq j_1, \ldots, j_s \leq m$, $1 \leq s < m$ such that $\cup_{\alpha=1}^q I_{i_\alpha} = \cup_{\alpha=1}^s J_{j_\alpha}$. Let $X \subseteq \mathbb{N}$ be the set outer full vertices, $X \cap J_l = \emptyset$, $l = 1, \ldots, m$. A partition $I \in \mathcal{P}(\cup_{l=1}^m J_l \cup X)$ is connected w.r.t the blocks $J_1, \ldots, J_m$ and the outer points $X = \{k_1, \ldots, k_n\}$ if $I \in \mathcal{P}_c(J_1, \ldots, J_m, \{k_1\}, \ldots, \{k_n\})$. We then write $I \in \mathcal{P}_c(J_1, \ldots, J_m; X)$.

A graph G is connected, if there exists an enumeration of its vertices such that each two subsequent vertices are connected by an edge. The set of connected generalized Feynman graphs with n outer full vertices and m inner full vertices is denoted by $\mathcal{F}_c(n, m)$.

Lemma 6.2. *A generalized Feynman graph $G \in \mathcal{F}(n, m)$ is connected if and only if the partition associated to G, cf. Lemma 3.3 and Fig. 1, is connected w.r.t. the blocks $J_1, \ldots, J_m$ of the legs of the inner full vertices and points X of the outer full vertices.*

Proof. As we can treat the points in $X = \{k_1, \ldots, k_n\}$ as n additional blocks $J_{m+1} = \{k_1\}, \ldots, J_{m+n} = \{k_n\}$, it suffices to prove the statement for $n = 0$.

Let $G \in \mathcal{F}_c(0, m)$ and $I = \{I_1, \ldots, I_k\}$ be the associated partition in $\mathcal{P}(\cup_{l=1}^m J_l)$. From Section 3 it is clear that there exists a bijection between the full vertices of G and the sets $\{J_1, \ldots J_m\}$ and between the empty vertices and the sets $\{I_1, \ldots, I_k\}$. Let $\{i_1, \ldots i_q\} \subseteq \{1, \ldots, k\}$ and $\{j_1, \ldots, j_s\} \subseteq \{1, \ldots, m\}$ such that $\cup_{\alpha=1}^q I_{i_\alpha} = \cup_{\alpha=1}^s J_{j_\alpha}$. Then all edges from inner full vertices associated to one J_j with index j in $\{j_1, \ldots, j_s\}$ go to an empty vertex associated with an I_i with $i \in \{i_1, \ldots, i_q\}$ and vice versa. Hence no edge leaves/comes into the subgraph $G' \subseteq G$ that consists out of the full vertices labelled by $\{j_1, \ldots j_s\}$ and the empty ones labelled by $\{i_1, \ldots i_q\}$ and all the edges between these vertices. By connectedness of G, $G' = G$. Hence $\{i_1, \ldots, i_q\} = \{1, \ldots, k\}$ and $\{j_1, \ldots, j_s\} = \{1, \ldots, m\}$.

Conversely, let $I \in \mathcal{P}_c(J_1, \ldots, J_m)$, $I = \{I_1, \ldots, I_k\}$, be connected and G the associated generalized Feynman graph and G' be a maximal connected subgraph of G. Let $\{i_1, \ldots i_q\}$ and $\{j_1, \ldots, j_s\}$ be the index sets of the empty respectively full vertices of G' obtained through the identification of full inner vertices with sets $\{J_1, \ldots, J_m\}$ and empty inner vertices with $\{I_1, \ldots I_k\}$. As G' is maximal, all edges in G connected to a vertex in G' are in G', hence $\cup_{\alpha=1}^q I_{i_\alpha} = \cup_{\alpha=1}^s J_{j_\alpha}$. From the connectedness of I it follows that $\{i_1, \ldots, i_q\} = \{1, \ldots, k\}$ and $\{j_1, \ldots, j_s\} = \{1, \ldots, m\}$. Hence $G' = G$. ∎

Proposition 6.3. *Let $J_1, \ldots, J_m \subseteq \mathbb{N}$ be disjoint sets and $\langle J_1 \cdots J_m \rangle_{\nu_0}^{(T)}$ be the truncated moment where each random variable $J_1, \ldots, J_m$ in the combinatorics of Def 3.1 is treated*

as one object. Then this "block truncated" moment has the following expansion into truncated moments $\langle \phi(x_1) \cdots \phi(x_n) \rangle_{\nu_0}^T$*:*

$$\langle J_1 \cdots J_m \rangle_{\nu_0}^{(T)} = \sum_{\substack{I \in \mathcal{P}_c(J_1,\ldots,J_m) \\ I=\{I_1,\ldots,I_k\}}} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^T . \tag{22}$$

Proof. Note that the ordinary moment functions determine the (block) truncated moments and vice versa. Hence, (22) holds if and only if the right hand side of this equation fulfills the defining equation for the left hand side, i.e. if and only if for all $m \in \mathbb{N}$

$$\langle J_1 \cdots J_m \rangle_{\nu_0} = \sum_{\substack{I \in \mathcal{P}\{1,\ldots,m\} \\ I=\{I_1,\ldots,I_k\}}} \prod_{l=1}^{k} \left[\sum_{\substack{Q_l \in \mathcal{P}_c(J_q : q \in I_l) \\ Q_l = \{Q_{l,1},\ldots Q_{l,k_l}\}}} \prod_{s_l=1}^{k_l} \langle Q_{l,s_l} \rangle_{\nu_0}^T \right] \tag{23}$$

holds. Given $\tilde{I} = \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, m\}$, we have introduced the notation $\mathcal{P}_c(J_q : q \in \tilde{I})$ for $\mathcal{P}_c(J_{i_1}, \ldots, J_{i_s})$.

On the other hand, we can expand the left hand side of (23) into truncated moment functions

$$\langle J_1 \cdots J_m \rangle_{\nu_0} = \sum_{\substack{R \in \mathcal{P}(\cup_{l=1}^m J_l) \\ R=\{R_1,\ldots,R_k\}}} \prod_{l=1}^{k} \langle R_l \rangle_{\nu_0}^T \tag{24}$$

and we have to prove that the right hand side of (23) equals the right hand side of (24).

Given $I \in \mathcal{P}\{1, \ldots, m\}$, $I = \{I_1, \ldots, I_k\}$ and $Q_l \in \mathcal{P}_c(J_q : q \in I_l)$ for $l = 1, \ldots, k$ one gets a partition $R = R(I, Q_1, \ldots, Q_k)$ from $\mathcal{P}(\cup_{l=1}^m J_l)$ setting $R = \cup_{l=1}^k Q_l$. The corresponding contributions to the right hand side of (23) and (24) are obviously equal. It remains to prove that the mapping $R(I, Q_1, \ldots, Q_k)$ from the index set of the total sum on the right hand side of (23) to $\mathcal{P}(\cup_{l=1}^m J_l)$ is one to one.

Again, this can be proven by construction of the inverse mapping. Let $R \in \mathcal{P}(\cup_{l=1}^m J_l)$ be given. For $1 \leq q < j \leq m$ we say that R connects q and j, in notation $q \sim_R j$, if the full inner vertices corresponding to J_q and J_j, respectively, are connected in the generalized Feynman graph corresponding to R, cf. Section 3. Obviously, $\sim_R$ is an equivalence relation on $\{1, \ldots, m\}$. Let $I = \{I_1, \ldots, I_k\}$ be the equivalence classes of $\sim_R$, then $I \in \mathcal{P}\{1, \ldots, m\}$. For $l = 1, \ldots, k$, let $Q_l = \{\tilde{R} \in R : \tilde{R} \subseteq \cup_{q \in I_l} J_q\}$. It remains to show that $Q_l \in \mathcal{P}_c(J_q : q \in I_l)$.

Firstly, $Q_l = \{Q_{l,1}, \ldots, Q_{l,k_l}\} \in \mathcal{P}(\cup_{q \in I_l} J_q)$. If not, then there are some points in $\cup_{q \in I_l} J_q$ that are not in $\cup_{s_l=1}^{k_l} Q_{l,s_l}$. A set $\tilde{R} \in R$ that contains at least one of these points, say from J_q for $q \in I_l$, can not contain any point from J_j, $j \notin I_l$, as this would imply that one can go in the graph corresponding to R from the full inner vertex J_q to J_j via the empty vertex $\tilde{R}$ in contradiction with $q \not\sim_R j$. Hence $\tilde{R} \in Q_l$, but this contradicts the assumption that $\tilde{R}$ contains at least one element $\notin \cup_{s_l=1}^{k_l} Q_{l,s_l}$.

Secondly, Q_l is a connected partition with respect to $J_q, q \in I_l$, as the subgraph with full inner vertices J_q, $q \in I_l$, and empty inner vertices Q_{l,s_l}, $s_l = 1, \ldots, k_l$, in the graph

associated to R by definition of $\sim_R$ is connected. An application of Lemma 6.2 therefore concludes the proof. ∎

Combination of Lemmas 6.1, 6.2 and Proposition 6.3 now gives the general linked cluster theorem:

Theorem 6.4. *The perturbations series of* $\log Z_\Lambda$ *for the energy density* $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \phi^p$ *only contains the connected generalized Feynman graphs, i.e. in the sense of formal power series one gets*

$$\log Z_\Lambda = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{G \in \mathcal{F}_c(0,m)} \mathcal{V}_\Lambda[G] \,. \tag{25}$$

Let $\mathcal{P}_c^{\mathrm{Wick}}(J_1, \ldots, J_m; X)$ be the intersection of $\mathcal{P}_c(J_1, \ldots, J_m; X)$ and $\mathcal{P}^{\mathrm{Wick}}(J_1, \ldots, J_m; X)$ and let $\mathcal{F}_c^{\mathrm{Wick}}(n, m)$ be the collection of connected generalized Feynman graphs without self-contractions at the inner full vertices. The generalization of Lemmas 6.1, 6.2 and Prop. 6.3 to the Wick ordered case is straight forward. One obtains the Wick ordered version of the general linked cluster theorem:

Corollary 6.5. *If one replaces* $v(\phi)$ *by its Wick ordered counterpart, (25) still holds if one restricts the sum on the right hand side to* $\mathcal{F}_c^{\mathrm{Wick}}(0, m)$.

The main application of linked cluster expansions in statistical mechanics is to prove the existence of the free energy density in the TD limit and to obtain an approximative formula for it. Here, for simplicity, we restrict to short range forces. The adequate formulation is as follows: Let ν_0 be a measure with exponential clustering, i.e. $\exists m_0 > 0$ such that for $X, Y \subseteq \mathbb{N}$, $|\langle XY \rangle_{\nu_0} - \langle X \rangle_{\nu_0} \langle Y \rangle_{\nu_0}| \leq D \exp\{-m_0 \, \underline{d}(X, Y)\}$ where D is a constant depending only[11] on $\sharp X$ and $\sharp Y$ and $\underline{d}(X, Y) = \min\{|x_j - y_l| : j \in X, l \in Y\}$ is the minimal distance between the points in X and Y. It is well-known, see e.g. [20], that this is equivalent with $|\langle XY \rangle_{\nu_0}^T| \leq D' \exp\{-m_0 \, \underline{d}(X, Y)\}$ for $D' = D'(\sharp X, \sharp Y)$ another constant. This is just a more precise statement of F2. For the convenience of the reader we give a proof of this statement in Appendix A.

The TD limit is to let $\Lambda \nearrow \mathbb{R}^d$ in the sense of Van Hove, cf [20, p.14] and Appendix B below.

Theorem 6.6. *Let* ν_0 *be a measure with exponential clustering and* $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \phi^p$ *the energy density. Then the perturbation series for the free energy density* f_Λ *converges in the sense of formal power series. The limit* $f = \lim_{\Lambda \nearrow \mathbb{R}^d} f_\Lambda$ *is given by*

$$f = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{G \in \mathcal{F}_c(0,m)} \mathcal{V}'[G] \tag{26}$$

where $\mathcal{V}'[G]$ *is obtained from the same Feynman rules as* $\mathcal{V}_\Lambda$, *cf. Theorem 3.5, with the only difference that the integration over one inner full vertex is omitted*[12] *and* Λ *in the remaining integrations is replaced by* $\mathbb{R}^d$.

[11] In the non uv-regular situation things are getting slightly more complicated, cf. Section 8.

[12] Note that by the translation invariance of ν_0, the result for $\mathcal{V}'$ does not depend on the argument $y_l \in \mathbb{R}^d$ or the choice $l = 1, \ldots, m$ of this inner full vertex.

If one Wick orders $v(\phi)$, (26) still holds for the sum on the right hand side restricted to graphs without self-contractions.

Proof. We have to prove that $\lim_{\Lambda \nearrow \mathbb{R}^d} \mathcal{V}_\Lambda[G]/|\Lambda| = \mathcal{V}'[G]$ for all $G \in \mathcal{F}_c(0,m)$. We thus have to prove that the integrand in the Feynman rules for G fulfills the conditions on $I(y_1,\dots,y_m)$ in Appendix B. Obviously, it is translation invariant. As the measures under discussion are uv-regularized, one can prove (41) for $B=0$.

Let y_1 and y_2 be the values attached to two inner full vertices. As G is connected, there is a path on G from y_1 to y_2 passing through at most $0 < m_1 < m-2$ inner full vertices and $0 < m_2 < m-1$ inner empty vertices. On the path from y_1 to y_2 there must be at least one of the m_1 steps from one inner full vertex to its successor vertex of the same kind that is $\geq |y_1 - y_2|/m_1$. Let n be the number of legs of the inner empty vertex that has been passed during this step. Then the n arguments of the corresponding truncated function can be divided into two groups with a minimal mutual distance of $|y_1 - y_2|/m_1 n$. By the cluster hypothesis this leads to a decay of the integrand $\leq C \exp\{-(m_0/m_1 n)|y_1 - y_2|\}$ for C sufficiently large.

Let now $y_1 = 0$ and $y_2,\dots,y_m$ be the remaining values assigned to the inner full vertices. Note that $\sum_{l=2}^m |y_l| \leq (m-1)\max\{|y_l| : L = 2,\dots,m\}$, hence the integrand of $\mathcal{V}_\Lambda(G)$ fulfills the estimate (41) for $M = m_0/(m^2\bar{n})$ where $\bar{n}$ is the maximal number of legs at an empty vertex in $\mathcal{F}(0,m)$, i.e. $\bar{n} = m\bar{p}$. ∎

It is easy to verify the exponential clustering for the models of Section 4 provided that $|g(x)| \leq D'' \exp\{-m_0|x|\}$, see (11) and also [1]. Hence Theorem 6.4 applies to these measures. This is also true for the two-dimensional models described in Theorem 5.5, as the proof of the above theorem can be easily adapted to the case where logarithmic divergences occur at coinciding points of the integrand, cf. Appendix B.

7 TD limit of (truncated) moment functions

In this short section we apply the results of Section 6 to the generating functionals of the truncated moment functions of the interacting measure ν_Λ in order to complete the solution of Problems 3 and 4 of Section 2. Apart from the input from Section 6, the methods we use here are more or less standard, see e.g. [6].

For $h \in \mathcal{S}$, let $v_h(\phi) = \sum_{p=0}^{\bar{p}} (\lambda_p - i\delta_{1,p}h)\phi^p$ be the energy density $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \phi^p$ with an additional "Schwinger term[13]" $-ih\phi$. Let $V_{\Lambda,h}(\phi) = \int_\Lambda v_h(\phi)\,dx$ and $Z_\Lambda(h) = \langle e^{-V_{\Lambda,h}}\rangle_{\nu_0}$. Obviously, $\mathcal{C}_{\nu_\Lambda}(h) = Z_\Lambda^{-1}\langle e^{i\langle\phi,h\rangle}e^{-V_\Lambda}\rangle_{\nu_0} = Z_\Lambda(h)/Z_\Lambda$ for $h \in \mathcal{S}$, $\mathrm{supp} h \subseteq \Lambda$. Hence, $\mathcal{C}^T_{\nu_\Lambda}(h) = \log Z_\Lambda(h) - \log Z_\Lambda$ and

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle^T_{\nu_\Lambda} = (-i)^n \left.\frac{\delta^n \log Z_\Lambda(h)}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\right|_{h=0} \quad \text{for } x_1,\dots,x_n \in \Lambda,\ n \in \mathbb{N}. \tag{27}$$

[13]Here the imaginary unit $i = \sqrt{-1}$ in front of the Schwinger term has been chosen in order to match with our conventions that the generating functional is the characteristic function, i.e. the (functional) Fourier transform and not the Laplace transform.

We want to find a graphical expression for (27). Let $\mathcal{F}_c^{\mathrm{Sw}\cdot}(m)$ be the collection of connected generalized Feynman graphs without outer vertices and with one additional type of inner full vertex (henceforth called Schwinger vertex) such that the total number of inner full vertices is m. The additional vertex type has one leg and corresponds to the Schwinger term. For $G \in \mathcal{F}_c^{\mathrm{Sw}\cdot}(m)$ and $h \in \mathcal{S}$ let $\mathcal{V}_\Lambda[G](h)$ be the value obtained according to the Feynman rules Def. 3.4 where $-ih$ is the coupling constant for the additional one-legged vertex. Then, by Theorem 6.4, $\log Z_\Lambda(h) = \sum_{m=1}^\infty \frac{(-1)^m}{m!} \sum_{G\in\mathcal{F}_c^{\mathrm{Sw}\cdot}(m)} \mathcal{V}_\Lambda[G](h)$ holds in the sense of power series in the formal parameters $\lambda_0, \ldots, \lambda_{\bar{p}}, h$. Inserting this into (27), one gets for $x_1, \ldots, x_n \in \Lambda$, $n \in \mathbb{N}$

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_{\nu_\Lambda}^T = (-i)^n \sum_{m=1}^\infty \frac{(-1)^m}{m!} \sum_{G\in\mathcal{F}_c^{\mathrm{Sw}\cdot}(m)} \frac{\delta^n \mathcal{V}_\Lambda[G](h)}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\bigg|_{h=0}. \tag{28}$$

For $G \in \mathcal{F}_c^{\mathrm{Sw}\cdot}(m)$ let n' be the number of the one-legged inner full vertices corresponding to the Schwinger term in the energy density. Then, $\frac{\delta^n \mathcal{V}_\Lambda[G](h)}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\Big|_{h=0} = 0$ if $n' \neq n$ and

$$\frac{\delta^n \mathcal{V}_\Lambda[G](h)}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\bigg|_{h=0} = (-i)^n \sum_{\sigma\in\mathrm{Perm}(n)} \mathcal{V}_\Lambda[G'](x_{\sigma_1}, \ldots, x_{\sigma_n}) \tag{29}$$

if $n' = n$ and $G' \in \mathcal{F}_c(n, m-n)$ is the graph obtained from G by replacing all Schwinger vertices with outer full vertices. $\mathrm{Perm}(n)$ is the permutation group of n objects. Obviously, all graphs from $\mathcal{F}_c(n, m-n)$ can be obtained in this way from some $G \in \mathcal{F}_c^{\mathrm{Sw}\cdot}(m)$.

For a generic $G \in \mathcal{F}_c^{\mathrm{Sw}\cdot}(m)$, each of its m inner full vertices can be a Schwinger vertex or not. If one has to choose exactly n from m vertices to become Schwinger vertices, there are thus $\binom{m}{n}$ possibilities. Once this choice has been done, every permutation of the n Schwinger vertices and the $m-n$ remaining inner full vertices leads to a distinct G' as full vertices are distinguishable. The covering $\{G \in \mathcal{F}_c^{\mathrm{Sw}\cdot} : G \text{ has } n \text{ Schwinger vertices}\} \ni G \to G' \in \mathcal{F}_c(n, m-n)$ thus is $\binom{m}{n}$-fold. Using this and inserting (29) into (28) one obtains

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_{\nu_\Lambda}^T = (-i)^{2n} \sum_{m=n}^\infty \frac{(-1)^m}{m!}\, n! \binom{m}{n} \sum_{G\in\mathcal{F}_c(n,m-n)} \mathcal{V}_\Lambda[G](x_1, \ldots, x_n). \tag{30}$$

The factor $n!$ stems from the sum over $\mathrm{Perm}(n)$ in (29). Re-arranging (30) in powers of the formal parameters $\lambda_0, \ldots, \lambda_{\bar{p}}$ then yields

Theorem 7.1. *For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \Lambda$ the truncated moment functions of the interacting measure ν_Λ are given by the formal power series*

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_{\nu_\Lambda}^T = \sum_{m=0}^\infty \frac{(-1)^m}{m!} \sum_{G\in\mathcal{F}_c(n,m)} \mathcal{V}_\Lambda[G](x_1, \ldots, x_n). \tag{31}$$

Let ν_0 have the exponential clustering property. Then, the right hand side of (31) converges to $\langle \phi(x_1)\cdots\phi(x_n)\rangle_\nu^T$ as $\Lambda \nearrow \mathbb{R}^d$ which is understood here as the formal power series given

on the r.h.s. of (31) with $\mathcal{V}_\Lambda[G](x_1,\dots,x_n)$ replaced by $\mathcal{V}[G](x_1,\dots,x_n)$. The latter expression is obtained through the Feynman rules as in Definition 3.4 with the integration over the inner full vertices extended over all $\mathbb{R}^d$.

Proof. Only the convergence in the TD limit $\mathcal{V}_\Lambda[G](x_1,\dots,x_n) \to \mathcal{V}[G](x_1,\dots,x_n)$ as $\Lambda \nearrow \mathbb{R}^d$ needs to be proven.

As G is connected and ν_0 is clustering exponentially fast, one can apply arguments similar to those in the proof of Theorem 6.6 to prove that the integrand in $\mathcal{V}_\Lambda[G](x_1,\dots,x_n)$is of exponential decay if any of the values attached to the inner full vertices becomes separated from any of the outer points $x_1,\dots,x_n \in \mathbb{R}^d$. Thus, the assertion of the theorem follows from Lebesgue's theorem of dominated convergence. ■

Let $\tilde{\mathcal{F}}(n,m)$ be the collection of generalized Feynman graphs with m inner full vertices and n outer full vertices such that any connected component of G contains at least one outer full vertex. Using Theorem 7.1 in combination with Def. 3.1 gives:

Corollary 7.2. *As a formal power series, the moment functions of the interacting measure $\nu = \lim_{\Lambda \nearrow \mathbb{R}^d} \nu_\Lambda$ are given by*

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle_\nu = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{G\in\tilde{\mathcal{F}}(n,m)} \mathcal{V}[G](x_1,\dots,x_n). \tag{32}$$

This completes the solution of Problem 4 in Section 2. Clearly, when replacing $v(\phi)$ with its Wick-ordered counterpart, Theorem 7.1 and Corollary 7.2 remain true if one restricts the sums on the right hand side of (31)and (32), respectively, to gen. Feynman graphs without self-contractions. An extension to the models described in Theorem 5.5 is also straight forward, cf. the last paragraph of Section 6.

8 Classical particles in the grand canonical ensemble

In this section we apply the results of the two preceding sections to the models of Section 4. We start with a summary of the physical interpretation of these models, see also [4, 5]:

Let in (9) be $z > 0$ and $a, \sigma^2 = 0$, i.e. $\psi(t)$ is purely Poisson. We again consider the measure ρ_0 associated with $\mathcal{C}_{\rho_0}(h) = \exp\{\int_{\mathbb{R}^d} \psi(h)\, dx\}$, $h \in \mathcal{S}$. The coordinate process η is a marked Poisson process with intensity z, where the mark space is $\mathbb{R}$ and the distribution of marks r. In other words, η has the interpretation of noninteracting classical, continuous particles in the configurational grand canonical ensemble with activity z, see e.g. [20], where each particle carries a r-distributed random charge with r as in (9). The random field $\phi = g * \eta$ obtained as the solution of $L\phi = \eta$ then has the natural interpretation as a static (short range) field associated to the charge distribution η. The interaction of the system of charged particles η can then be defined as $U_\Lambda(\eta) = V_\Lambda(g * \eta)$ with $V_\Lambda(\phi) = \int_\Lambda v(\phi)\, dx$ where we have tacitly uv-regularized the kernel $g = g_\epsilon$ which implies that the random field ϕ has continuous paths, or, equivalently that $\nu_0 = \nu_0^\epsilon$, the probability measure associated with ϕ, fulfills property 1 of Section 2.

The grand canonical partition function is defined as $\Xi_\Lambda = \langle e^{-\beta U_\Lambda}\rangle_{\rho_0}$ with $\beta = \frac{1}{k_B T}$ the inverse temperature, k_B is Boltzmann's constant. Note that ν_0 is the image measure of ρ_0 under the mapping $\mathcal{S}' \ni \eta \to \phi = g * \eta \in \mathcal{S}'$. By the transformation formula of measures

$$\Xi_\Lambda = \langle e^{-\beta U_\Lambda}\rangle_{\rho_0} = \langle e^{-\beta V_\Lambda}\rangle_{\nu_0} = Z_\Lambda. \tag{33}$$

Hence, for $v(\phi) = \sum_{p=0}^{\bar{p}} \lambda_p \phi^p$, the expansion obtained in Theorem 6.6 in combination with the Feynman rules Theorem 4.1 is valid for $\beta p(\beta, z) = \lim_{\Lambda \nearrow \mathbb{R}^d} \log \Xi_\Lambda(\beta)/|\Lambda|$ where $p(\beta, z)$ is the pressure function, cf. [20, Theorem 3.4.6.].

Furthermore, let $d\rho_\Lambda(\eta) = \Xi_\Lambda^{-1} e^{-\beta U_\Lambda(\eta)} d\rho_0(\eta)$ be the interacting grand canonical measure, then the transformation formula yields

$$\langle \eta(x_1)\cdots\eta(x_n)\rangle_{\rho_\Lambda} = \langle (L\phi)(x_1)\cdots(L\phi)(x_n)\rangle_{\nu_\Lambda} = L^{\otimes n}\langle \phi(x_1)\cdots\phi(x_n)\rangle_{\nu_\Lambda}. \tag{34}$$

This obviously implies $\langle \eta(x_1)\cdots\eta(x_n)\rangle^T_{\rho_\Lambda} = L^{\otimes n}\langle \phi(x_1)\cdots\phi(x_n)\rangle^T_{\nu_\Lambda}$. Summarizing the above discussion, we get

Theorem 8.1. *The expansions obtained for the (truncated) moment functions in Theorem 7.1 and Corollary 7.2 holds also for the (truncated) moments of the interacting grand-canonical measure ρ_Λ if $\mathcal{V}_\Lambda[G]$ defined in Theorem 4.1 is modified in the sense that for an edge connecting an inner empty and an outer full vertex $\overset{x}{\times}\!\!-\!\!\circ\overset{z}{}$ the propagator function $g(x-z)$ is replaced with a Dirac delta function[14] $\delta(x-z)$.*

In particular, the TD-limit of the (truncated) moments $\langle \eta(x_1)\cdots\eta(x_n)\rangle^T_\rho = \lim_{\Lambda \nearrow \mathbb{R}^d} \langle \eta(x_1)\cdots\eta(x_n)\rangle^T_{\rho_\Lambda}$ and $\langle \eta(x_1)\cdots\eta(x_n)\rangle_\rho = \lim_{\Lambda \nearrow \mathbb{R}^d} \langle \eta(x_1)\cdots\eta(x_n)\rangle_{\rho_\Lambda}$ exists in the sense of formal power series. Furthermore, the pressure function $p(\beta, z)$ in the sense of formal power series is given by $k_B T$ times the right hand side of (26).

Let us go one step further and consider the case where in (9) $z > 0$ and $\sigma^2 > 0$. There is a Gaussian and a (marked) Poisson contribution to the random field η. While the Poisson contribution is interpreted as grand canonic ensemble of mesoscopic charged particles, the Gaussian contribution can be interpreted as a white noise fluctuation of the charge density due to microscopic particles.[15] The random field η now stands for the total random charge distribution containing the mesosopic and the microscopic part. The above analysis can be repeated word by word and Theorem 8.1 also gives the expansions of the pressure and the (truncated) moment functions of the given mixed system containing two clearly separated scales. It is also clear, that there is Wick-ordered version of Theorem 8.1.

[14] Even though there is some similarity, the Feynman rules in this theorem should not be mixed up with the Feynman rules for the amputated Green's functions in the calculation of effective actions in the renormalization group [11].

[15] In fact, the Gaussian part can be seen as the scaling limit of a Poisson contribution, η_z, neutral in average, where the intensity $z \to \infty$ and the charges are being scaled $\sim 1/\sqrt{z}$. I.e. in (9) we take $\sigma^2 = 0, a = 0$ and r, fulfilling $c_1 = 0$, is replaced with $r_z(A) = r(\sqrt{z}A)$ for $A \subseteq \mathbb{R}^d$ measurable. Taking the limit $\lim_{z\to\infty} \psi_z(t) = c_2 t^2/2$ implies that η_z converges in law to a Gaussian white noise as $z \to \infty$, cf. [4, 5] for the details.

Having set the frame, we want to do calculations for some specific examples, where the diagrammatic structure is particularly simple. This is e.g. the case, when the measure ν_0 is symmetric and all inner empty vertices with an odd number of legs vanish, cf. Corollary 3.7. In the given situation, this can be achieved choosing the charge distribution r of the non-interacting gas symmetric, $r(-A) = r(A)\ \forall A \subseteq \mathbb{R}$ measurable, and $a = 0$ which implies $c_n = 0$ for odd $n \in 2\mathbb{N}+1$. Furthermore, the simplest non-trivial kind of interaction is $v(\phi) = \lambda_2\phi^2$ for $\lambda_2 > 0$. Here we do not use Wick-ordering as it is more difficult to interpret and the only term it removes in the expansion of the free energy density is the first order contribution, which is easy to calculate.

To understand this interaction, let $\eta = \sum_{l=1}^{n} s_l\delta_{y_l}$ be a finite, discrete charge distribution and $U(\eta) = V(g * \eta) = \int_{\mathbb{R}^d} v(\phi)\,dx$ is the potential energy without cut-offs. One obtains

$$U(\eta) = \lambda_2 \int_{\mathbb{R}^d} \left(\sum_{l=1}^{n} s_l\, g(y_l - y) \right)^2 dy = \sum_{j,l=1}^{n} s_j s_l\, \tilde{g}_1(y_j - y_l) + \sum_{l=1}^{n} s_l^2 \tilde{g}_1(0), \tag{35}$$

where $\tilde{g}_1 = \lambda_2\, g * g$. The second sum on the right hand side of (35) can be seen as self energy term or a (negative) chemical potential that depends on the charge s of the particle. It can be removed by an adaptation of z and r.[16] The first sum is a usual pair interaction potential for charged particles. A ϕ^p interaction would also contain l-body potentials for $l \leq p$.

We want to calculate the free energy density $f = \beta p(z, \beta)$ for small β and z (low density high temperature regime). In the diagrammatic expansion given in Theorem 6.6, only two-legged interaction vertices appear. Like in Fig. 3 in Section 4, we can introduce a new type of edge denoted by a thin line  and we get that the gen. Feynman graphs of m-th order are exactly all graphs with an arbitrary number of indistinguishable inner empty vertices with an arbitrary number of indistinguishable legs and exactly m "thin" edges connecting two inner empty vertices. We note that, as the legs of  are distinguishable, the thin edge has to be treated as a directed edge in order to get the right multiplicity of a given graph. The evaluation rules $\mathcal{V}'[G]$ for a graph of this new type are simply to replace each thin edge by $\tilde{g}_1$ and to multiply with c_n for each inner empty vertex with n legs. Then one integrates over all but one of the inner empty vertices. That this description in fact gives the right rules, i.e. that the infra-red cut-off Λ in the TD limit can be shifted from the integration over the inner full vertices to the inner empty vertices, follows from the argument of Appendix B.

Figure 6 shows the graphs G that are contributing to the free energy density up to fourth order together with their multiplicity and value $\mathcal{V}'[G]$. $\tilde{g}_n = \tilde{g}_1^{*n}$ is the n-fold convolution of $\tilde{g}_1$ with itself.

Let us consider a simple example in $d = 2$ dimensions with only two kinds of charge $\pm c$, i.e. $c_n = \delta_{2,n}\sigma^2 + c^n z/2$, σ being the intensity of the Gaussian background, and $L =$

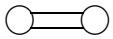
[16] Take e.g. the simplest case where $r = (\delta_c + \delta_{-c})/2$ and $s_l^2 \equiv c^2 > 0$. As $z = (2\pi M/\beta)^{d/2} e^{\beta\mu}$ with $M > 0$ the mass of the particles (assumed to be equal for particles with positive and negative charge) and μ the chemical potential, one can compensate the self energy term by replacing μ with $\mu + c^2\tilde{g}_1(0)$.

1) 

$m = 1,\ u = 1$
$\mathcal{V}' = c_2 \tilde{g}_1(0)$

2) 

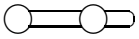
$m = 2,\ u = 1$
$\mathcal{V}' = c_4 \tilde{g}_1(0)^2$

3) 

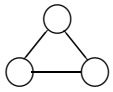
$m = 2,\ u = 2$
$\mathcal{V}' = c_2^2 \tilde{g}_2(0)$

4) 

$m = 3,\ u = 1$
$\mathcal{V}' = c_6 \tilde{g}_1(0)^3$

5) 

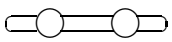
$m = 3,\ u = 12$
$\mathcal{V}' = c_4 c_2 \tilde{g}_2(0) \tilde{g}_1(0)$

6) 

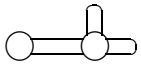
$m = 3,\ u = 8$
$\mathcal{V}' = c_2^3 \tilde{g}_3(0)$

7) 

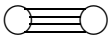
$m = 4,\ u = 1$
$\mathcal{V}' = c_8 \tilde{g}_1(0)^4$

8) 

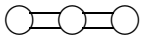
$m = 4,\ u = 24$
$\mathcal{V}' = c_4^2 \tilde{g}_2(0) \tilde{g}_1(0)^2$

9) 

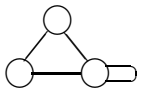
$m = 4,\ u = 24$
$\mathcal{V}' = c_6 c_2 \tilde{g}_2(0) \tilde{g}_1(0)^2$

10) 

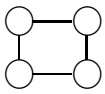
$m = 4,\ u = 8$
$\mathcal{V}' = c_4^2 \int_{\mathbb{R}^d} \tilde{g}_1^4 \, dx$

11) 

$m = 4,\ u = 48$
$\mathcal{V}' = c_4 c_2^2 \tilde{g}_2(0)^2$

12) 

$m = 4,\ u = 96$
$\mathcal{V}' = c_2^2 c_4 \tilde{g}_3(0) \tilde{g}_1(0)$

13) 

$m = 4,\ u = 48$
$\mathcal{V}' = c_2^4 \tilde{g}_4(0)$

Figure 6: Graphs for the gas of charged particles, neutral in average, with pair interaction up to fourth order. m =order, u =multiplicity, $\mathcal{V}'$=value.

$(-\Delta + m_0^2)$ with m_0 the range of the interaction. We get that $g(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \frac{1}{(|k|^2+m_0^2)} \times e^{ik\cdot x}\, dk$ diverges logarithmically at 0. The measure ν_0 thus does not fulfill property 1 of Section 2. In fact, the Poisson contribution to the random field η associated to ρ_0 has discrete support and the random field $\phi = g * \eta$ has singularities on the support of the Poisson part of η. However, by choosing the interaction to be ϕ^2, we see that only $\tilde{g}_1 = \lambda_2\, g * g$ enters into the perturbation series which is a continuous function for $d < 4$. The uv-cut-offs therefore can be removed from the perturbation series.

The perturbation coefficients up to forth order can now be calculated explicitly by solution of rather elementary integrals:

$$\tilde{g}_n(0) = \lambda_2^n (2\pi)^{-1} \int_0^\infty \frac{a\, da}{(a^2+m_0^2)^{2n}} = \lambda_2^n \frac{m_0^{2-4n}}{2\pi(4n-2)} \tag{36}$$

and

$$\int_{\mathbb{R}^d} \tilde{g}_1^4 dx = \lambda_2^4 (2\pi)^{-6} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{((|k-q|^2+m_0^2)^2(|q|^2+m_0^2)^2}\, dq \right]^2 dk = \lambda_2^4 \frac{m_0^{-6}}{64\pi^3}. \tag{37}$$

This gives the following equation of state:

$$\begin{aligned} p(z,\sigma,c,\beta,\lambda_2,m_0) &= -\lambda_2 \frac{\sigma^2+\frac{z}{2}c^2}{4m_0^2\pi} + \frac{\lambda_2^2\beta}{2}\left(\frac{zc^4}{32m_0^4\pi^2} + \frac{(\sigma^2+\frac{z}{2}c^2)^2}{6m_0^6\pi}\right) \\ &- \frac{\lambda_2^3\beta^2}{6}\left(\frac{zc^6}{128m_0^6\pi^3} + \frac{(s^2+\frac{z}{2}c^2)zc^4}{8m_0^8\pi^2} + \frac{2(\sigma^2+\frac{z}{2}c^2)^3}{5m_0^{10}\pi}\right) \\ &+ \frac{\lambda_2^4\beta^3}{24}\left(\frac{zc^8}{512m_0^8\pi^4} + \frac{z^2c^8}{32m_0^{10}\pi^3} + \frac{(\sigma^2+\frac{z}{2}c^2)zc^6}{16m_0^{10}\pi^3} + \frac{z^2c^8}{32m_0^6\pi^3}\right. \\ &+ \left.\frac{(\sigma^2+\frac{z}{2}c^2)^2zc^4}{6m_0^{12}\pi^2} + \frac{3(\sigma^2+\frac{z}{2}c^2)^2zc^4}{10m_0^{12}\pi^2} + \frac{12(\sigma^2+\frac{z}{2}c^2)^4}{7m_0^{14}\pi}\right) + \mathcal{O}(\lambda_2^5). \end{aligned} \tag{38}$$

Remark 8.2. The graphs that one obtains for the ϕ^2-interaction are obviously very similar to those of the Mayer series [18, 20, 27]. If there is only one type of particles with charge c, the main difference (neglecting combinatorial matters) is that in the Mayer series edges are evaluated with the Mayer function $w(y_1-y_2) = e^{-\beta c^2 \tilde{g}_1(y_1-y_2)} - 1$ instead of $\tilde{g}_1(y_1-y_2)$, which of course is a big advantage if the two point potential $\tilde{g}_1$ has a (repulsive) singularity at zero as e.g. in the case of the Lennard-Jones potential. In such cases, the perturbation expansion given in this article becomes plagued by very non-trivial uv-singularities but the Mayer series is not. This is the reason why the perturbetion expansion, though in principle known to physicists, see e.g. [27, Sect. 3.3, Eq. 42], is not particularly popular. The Mayer series for the gas of particles with two types of charges $\pm c$ however also contains the Mayer function $w(y_1-y_2,+,-) = e^{+\beta c^2 \tilde{g}_1(y_1-y_2)} - 1$ for the interaction of a + charge with a − charge which is more singular than $\tilde{g}_1(x-y)$. Also, the analytic calculation in low orders of the graphs including "propagators" $\tilde{g}_1(y_1-y_2)$ seems to be more easy than for the propagators $w(y_1-y_2)$. In some particular situations, there might therefore be some physical interest in the derived series expansion, even

though it is not the objective of this article to solve a concrete physical problem in the thermodynamics of gases, fluids or electrolytes. ∎

At the end of this section, we want to give some brief and non technical remarks on the uv-problem, leaving most of the work for the future. If the measure ν_0 from Section 4 is not Gaussian and $\tilde{g}_1(y)$ for $y \to 0$ has an algebraic singularity $\sim |y|^{-\theta}$, $\theta > 0$, already the ϕ^2-perturbation series is not power-counting renormalizable: If one e.g. considers graphs of the kind 3) and 10) in Fig. 6 for an arbitrary (even) number m of legs, one gets the suspicious degree of divergence $m\theta - d$.

Nevertheless, if one takes a look (35) in the special situation where there is only one type of particle with charge $c > 0$, one can see that a simple 1st order local counterterm $\lambda_1^\epsilon \phi$ (i.e. a chemical potential) with $\lambda_1 = -c g_1^\epsilon(0) / \int_{\mathbb{R}^d} g \, dx$ removes all singularities in the limit $\epsilon \searrow 0$. This is in striking contrast with the non-renormalizability.

We say that a graph has a self-contraction of the second kind if a subgraph  occurs, cf. the graphs 1), 2), 4), 5), 7), 8), 9) and 12) of Fig. 6. If one includes the above counter term into the perturbation series for the system with only one particle species, one can prove that all self-contractions of 2nd kind are being removed from the series. In fact, for each such graph, there is exactly one other graph where the self-contraction of the 2nd kind is replaced by  , with the interaction vertex of the linear counterterm. It is therefore clear that self-contractions of 2nd kind are caused by the self-energy terms on the right hand side of (35).

This observation has two immediate consequences: Firstly, in the case where $\tilde{g}_1(y)$ has an algebraic singularity at 0, the perturbation series remains non-power counting renormalizable, even though it can be "summed up" and then gives a finite result [20].

Secondly, if the singularity of $\tilde{g}_1(y)$ at $y = 0$ is only logarithmic, the self-contractions of 2nd kind are the only source of divergences[17], cf. the proof of Theorem 5.5 for the uv-finiteness of 2nd-self-contraction free graphs. The given choice of the counterterm removes the uv-divergences from the perturbation series. This e.g. occurs in the cases $d = 4$ and $L = (-\Delta + m_0^2)$, $d = 2$ and $L = (-\Delta + m_0^2)^{1/2}$ or $d = 2$, $L = (-\Delta + m_0^2)$ and $V_\Lambda(\phi) = \lambda_2 \int_\Lambda |\nabla \phi|^2 \, dx$ is of gradient type leading to a pair potential $\tilde{g}_1(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \frac{|k|^2}{(|k|^2 + m^2)^2} \, e^{ik \cdot x} \, dk$ with equally strong repulsive and attractive parts, i.e. $\int_{\mathbb{R}^2} \tilde{g}_1 \, dx = 0$.

A Facts about truncation

For the convenience of the reader, we give proofs of the well-known facts F1) – F4) on the combinatorics of truncation starting with F4):

Lemma A.1. *Let $u : C \to \mathbb{C}$ for C an open set in $\mathcal{S}$ be infinitely often partial differentiable and let $w : \mathbb{C} \mapsto \mathbb{C}$ be analytic on an open neighborhood of $u(C)$. Then for $\{h_n\}_{n \in \mathbb{N}} \subseteq \mathcal{S}$*

[17] The situation has some similarity with Gaussian ϕ^4-theory in $d = 3$ dimensions (take $L = (-\Delta + m_0^2)^{1/2}$), where there is also just one subgraph (see Fig. 3) causing logarithmic divergences.

and $J \subseteq \mathbb{N}$ *finite*

$$\partial_J w \circ u = \sum_{k=1}^{\sharp J} w^{(k)} \circ u \sum_{\substack{I \in \mathcal{P}(J) \\ I=\{I_1,\dots,I_k\}}} \prod_{l=1}^{k} [\partial_{I_l} u] \tag{39}$$

holds on C. *Here* $\partial_A = \left[\prod_{j\in A} \frac{\partial}{\partial h_j}\right]$ *for* $A \subseteq \mathbb{N}$ *finite and* $w^{(k)}(z) = \left(\frac{d^k}{dz^k} w\right)(z)$.

The proof is by use of Leibnitz' chain rule and induction over $\sharp J$, details can be found in [1, Lemma 3.3]. The application of this generalized chain rule to $J = \{1,\dots,n\}$, $u = \mathcal{C}^T$ ($\Rightarrow \mathcal{C}^T(0) = 0$) and w the exponential function, evaluation at $0 \in \mathcal{S}$ and doing the limit $h_l \to \delta_{x_l}$, $l \in J$, establishes fact F4. We note that in Lemma A.1 one can also replace $\mathcal{S}$ with $\mathbb{R}_+ = [0,\infty)$ and do right derivatives at zero for u infinitely often right differentiable, as required in Eq. (20). F1) is immediate from F4).

We prove F2) in the form required in Section 6. Let $X, Y \subseteq \mathbb{N}$ disjoint, then

$$\langle XY\rangle_{\nu_0} - \langle X\rangle_{\nu_0}\langle Y\rangle_{\nu_0} = \sum_{\substack{I \in \mathcal{P}_c(X,Y) \\ I=\{I_1,\dots,I_k\}}} \prod_{l=1}^{k} \langle I_l\rangle^T_{\nu_0}. \tag{40}$$

As $I \in \mathcal{P}_c(X,Y)$ there exists at least one $l = 1,\dots,k$ such that $X_l = I_l \cap X \neq 0$ and $Y_l = I_l \cap Y \neq 0$. If the truncated moment functions vanish exponentially for large separation of their arguments, we get that each term in the sum on the right hand side contains at least factor $|\langle I_l\rangle^T| \leq D' \exp\{-m_0 \underline{d}(X_l, Y_l)\} \leq D' \exp\{-m_0 \underline{d}(X,Y)\}$. The right hand side thus vanishes exponentially for large separation of X and Y.

Conversely, let ν_0 have the exponential clustering property. We proceed by induction over $n = \sharp X + \sharp Y$. If we set $\langle\emptyset\rangle^T_{\nu_0} = 0$ and $d(X,\emptyset) = 0$, the assertion $|\langle XY\rangle^T_{\nu_0}| \leq D' \exp\{-m_0 \underline{d}(X,Y)\}$ is trivial for $n = 0$. Suppose that it holds up to $n-1$, then each term on the right hand side of (40) except for the term $I = \{X \cup Y\}$ contains at least one factor on which the induction hypothesis applies and which thus vanishes exponentially fast as X and Y get separated. As the left hand side also vanishes exponentially fast, this must also apply to this remaining term $I = \{X \cup Y\}$. Hence F2) holds.

To get F3), consider Eq. (6). If n is odd, for each partition $I = \{I_1,\dots,I_k\}$ on the right hand side there exists at least one $l \in \{1,\dots,k\}$ such that $\sharp I_l$ is odd. Hence the vanishing of $\langle J\rangle^T_{\nu_0}$ for $J \subseteq \mathbb{N}$ with $\sharp J$ odd implies the vanishing of the left hand side of (6).

Let conversely the odd moments of ν_0 be vanishing. We proceed by induction over l and let $\sharp X = 2l+1$. For $l = 0$ we get $\langle X\rangle_{\nu_0} = \langle X\rangle^T_{\nu_0} = 0$. Suppose that $\langle X\rangle^T_{\nu_0} = 0$ for odd $\sharp X < 2l+1 = n$. Hence all term on the right hand side of (6) except for the one with $I = \{\{1,\dots,n\}\}$ vanish. But the left hand side is zero, and this remaining term therefore must be zero, too.

B TD limit for certain integrals

Let $I(y_1,\dots,y_m)$ be a translation invariant function such that

$$|I(y_1,y_2,\dots,y_m)| \;\le\; C\left(1+B\sum_{\substack{l,j=1\\ l\neq j}}^{m} 1_{\{|y_l-y_j|<1\}}(y_j-y_l)|\log|y_l-y_j||^n\right) \times \exp\{-M\sum_{l=2}^{m}|y_l-y_1|\} \tag{41}$$

for some $n\in\mathbb{N}$ and $M,B,C>0$. 1_A stands for the indicator function of the set A. Then the following holds in the TD limit:

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\frac{1}{|\Lambda|}\int_{\Lambda^m} I(y_1,\dots,y_m)\,dy_1\cdots dy_m = \int_{\mathbb{R}^{d(m-1)}} I(0,y_2,\dots,y_m)\,dy_2\cdots dy_m. \tag{42}$$

In fact, by Fubini's theorem and translation invariance

$$\frac{1}{|\Lambda|}\int_{\Lambda^m} I(y_1,\dots,y_m)\,dy_1\cdots dy_m = \frac{1}{|\Lambda|}\int_{\Lambda_{y_1}}\left[\int_{\Lambda_{y_1}^{m-1}} I(0,y_2,\dots,y_m)\,dy_2\cdots dy_m\right]dy_1, \tag{43}$$

where $\Lambda_{y_1}=\Lambda-y_1$. We consider the expression in the brackets $[\cdots]$ as a function of y_1 and we obtain

$$\begin{aligned}[\cdots] \;&=\; \int_{\mathbb{R}^{d(m-1)}} I(0,y_2,\dots,y_m)\,dy_1\cdots dy_m \\ &-\; \sum_{l=0}^{m-2}(-1)^{l+1}\int_{\mathbb{R}^{dl}}\int_{(\mathbb{R}^d\setminus\Lambda_{y_1})}\int_{\Lambda_{y_1}^{m-2-l}} I(0,y_2,\dots,y_m)\,dy_2\cdots dy_m.\end{aligned} \tag{44}$$

We want to get an estimate for the sum on the right hand side of (44): Using (41) one obtains

$$\begin{aligned}&\left|\sum_{l=0}^{m-2}(-1)^{l+1}\int_{\mathbb{R}^{dl}}\int_{(\mathbb{R}^d\setminus\Lambda_{y_1})}\int_{\Lambda_{y_1}^{m-2-l}} I(0,y_2,\dots,y_m)\,dy_2\cdots dy_m\right| \\ \le\;& C(m-1)\left(\int_{\mathbb{R}^d} e^{-M|y|}dy + Bm\int_{\{|y|\le 1\}}|\log|y||^n\,dy\right)^{m-2} u(y_1,\Lambda)\end{aligned} \tag{45}$$

where $u(y_1,\Lambda)=\int_{\mathbb{R}^d\setminus\Lambda} e^{-M|y-y_1|}(1+B1_{\{|y-y_1|<1\}}|\log|y-y_1||^n)\,dy$. Hence, for $C'>0$ large enough,

$$\begin{aligned}&\left|\frac{1}{|\Lambda|}\int_{\Lambda^m} I(y_1,\dots,y_m)\,dy_1\cdots dy_m\right. \\ &\qquad\left.-\int_{\mathbb{R}^{d(m-1)}} I(0,y_2,\dots,y_m)\,dy_2\cdots dy_m\right| \le C'\frac{1}{|\Lambda|}\int_{\Lambda} u(y_1,\Lambda)\,dy_1\end{aligned} \tag{46}$$

We note that $u(y_1,\Lambda) \leq C'' e^{-\frac{M}{2}\underline{d}(\partial\Lambda,y_1)}$ for $y_1 \in \Lambda$ where $\underline{d}(\partial\Lambda, y_1)$ stands for the distance from y_1 to the boundary of Λ and $C'' = \int_{\mathbb{R}^d} e^{-\frac{M}{2}|y|}(1 + B|\log|y||^n)\,dy$. Let $\partial_a\Lambda = \{y \in \Lambda : \underline{d}(\partial\Lambda, y_1) < a\}$, then

$$\frac{1}{|\Lambda|}\int_\Lambda u(y_1,\Lambda)\,dy_1 \leq \frac{C''}{|\Lambda|}\left(e^{-Ma/2}|\Lambda \setminus \partial_a\Lambda| + |\partial_a\Lambda|\right) \leq C''(e^{-Ma/2} + |\partial_a\Lambda|/|\Lambda|). \qquad (47)$$

holds for all $a > 0$. As convergence $\Lambda \nearrow \mathbb{R}^d$ in the sense of Van Hove means that $|\partial_a\Lambda|/|\Lambda| \to 0\ \forall a > 0$ in the TD limit, the right hand side of (47) and hence (46) can be made arbitrarily small for Λ in the TD limit sufficiently large. This proves equation (42).

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